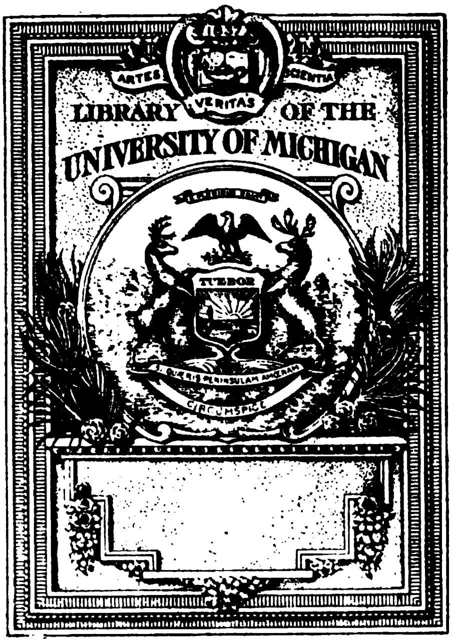
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Alexander Ziwet

AN

ELEMENTARY TREATISE

ON THE

DIFFERENTIAL CALCULUS

FOUNDED ON THE

METHOD OF RATES OR FLUXIONS

BY

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FOURTH EDITION REVISED AND CORRECTED

NEW YORK:

JOHN WILEY & SONS,

15 ASTOR PLACE.

1888.

New York: J. J. Little & Co., Printers,
10 to 20 Astor Place.

Gift
Prof. Alexander Ziwet
7/10-29

PREFACE.

THE difficulties usually encountered on beginning the study of the Differential Calculus, when the fundamental idea employed is that of infinitesimals or that of limits, together with the objectionable use of infinite series involved in Lagrange's method of derived functions, have induced several writers on this subject to return to the employment of Newton's conception of rates or fluxions. The readiness with which this conception is grasped, and the precision it gives to the preliminary definitions, promise an advantage which, however, is in most cases sacrificed by resorting to the use of limits in deducing the formulas for differentiation.

These considerations induced the authors of this work to seek to derive the differentials of the functions by an analytical method founded upon the notion of rates, and entirely independent of the difficult conceptions of infinitesimal increments and of limiting ratios.

The investigation thus initiated resulted in a satisfactory method of obtaining the differentials of the simple functions, which was embodied in a paper communicated to the *American Academy of Arts and Sciences*, January 14, 1873, by Professor J. M. Peirce, and published in the Proceedings of the Society. This paper was subsequently rewritten and published as a pamphlet.*

A complete résumé of the original paper, by J. W. L. Glaisher, of Trinity College, Cambridge, was published in August, 1874, in the *Messenger of Mathematics*, Macmillan & Co., London; vol. iv, page 58.

The method alluded to may be briefly described thus :—Denoting an assumed finite interval of time by dt, dx is so defined that $\frac{dx}{dt}$ is the

* *On a new Method of obtaining the Differentials of Functions, with especial reference to the Newtonian conception of rates or velocities*, by J. Minot Rice and W. Woolsey Johnson. D. Van Nostrand, Publisher, New York, 1875.

expression for the rate of x (page 12); and, after establishing a few elementary propositions, which are immediate consequences of the definitions, it is shown (page 17) that, when y is a function of x, $\frac{dy}{dx}$ has a value independent of the assumed finite value of dx. For the mode in which the differentials may be directly deduced, with the aid of this important proposition, the reader is referred to page 23 and to page 39.

It is not the intention of the authors to disparage the use of the limit as an instrument of mathematical research. It is only claimed that the difficulties attending the employment of this notion are so great as to render it desirable to avoid introducing it into the fundamental definitions of a subject necessarily involving many other conceptions new to the student.

The distinction between the view of the differential calculus here presented, and that found in most of the standard works on the subject hitherto published, may be stated thus:—The derivative $\frac{dy}{dx}$ is usually defined as the limit which the ratio of the finite quantities Δy and Δx approaches when these quantities are indefinitely diminished: when this definition is employed, it is necessary, before proceeding to kinematical applications, to prove that this limit is the measure of the relative rates of x and y. In this work the order is reversed; that is, dx and dy are so defined that their ratio is equal to the ratio of the relative rates of x and y, and in Chapter XI, by applying the usual method of evaluating indeterminate forms, it is shown that the limit of $\frac{\Delta y}{\Delta x}$, when Δx is diminished indefinitely, is equal to the ratio $\frac{dy}{dx}$. Thus the employment of limits is put off until we are prepared to show that the limit has a definite value capable of expression in a language already familiar to the student.

The early introduction of elementary examples of a kinematical character (see pp. 28, 37, and 57; also, Section X, p. 76) which this mode of presenting the subject permits, will be found to serve an important purpose in illustrating the nature and use of the symbols employed.

The method of treating limits employed in Section XXXVIII is a

modification of the method given in M. J. Bertrand's *Traité de Calcul différentiel et intégral*, pages 41 and 136, edition of 1864. To this valuable work we are also indebted for many other suggestions.

The determination of the maxima and minima values of functions, and the evaluation of indeterminate forms, are so treated as to be independent of Taylor's theorem. In the investigation of these subjects, and in many other cases, we have found it desirable to substitute for the demonstrations commonly given others more in harmony with the conception of rates.

It has been found necessary to devote an unusually large amount of space (from page 230 to page 415) to the geometrical applications of the differential calculus, in consequence of the lack of available text-books on Curve Tracing and on Higher Plane Curves. These pages include Chapter IX, which consists of a brief discussion of the equations and many of the properties of the best-known higher plane curves, and is introduced chiefly for convenience of reference. We trust this chapter will be found a very useful feature of the book.

To facilitate the use of this work as a text-book, it has been divided into short sections, each of which is followed by a copious collection of examples. In the arrangement of these examples the order of subjects in the section has been generally followed, and easy examples usually precede the more difficult ones.

It will not in general be found advisable for the beginner to solve all the examples on reading this work for the first time. They occupy nearly one third of the entire volume, and are intended as a collection from which the instructor may select at his discretion.

Many of these examples were prepared especially for this work. The others were taken chiefly from the collections of *Gregory* (Walton's edition), *Frenet*, and *Tisserand;* and from the treatises of *Todhunter*, *Williamson*, and *Connell.*

The first two pages of the Table of Contents, which relate to the differentiation of functions, are omitted.

A Limited Course.

When the time allotted to the Differential Calculus is insufficient for a more extended course, the following articles may be taken. The text of these articles occupies only about 140 pages. This course may be supplemented by selections from Chapter IX. Appropriate examples will be found at the beginning of the collections which accompany each section.

Art. 1–81, 93–101, 107–112, 119–139, 147–155, 159, 161–167, 200–209, 241–245, 311–326, 332–341, 353–357, 389–394, 398–404, inclusive.

ANNAPOLIS, MARYLAND,
September, 1879.

J. M. R.
W. W. J.

Note to the Third Edition.

An abridged edition of this work was published, August, 1880, which comprises the above mentioned limited course and some additional matter. The abridgment can be used in connection with Professor Johnson's Integral Calculus, which is published simultaneously with the the edition of the larger work.

The authors improve this opportunity to tender their thanks to those friends who have kindly suggested corrections and emendations which are embodied in the present edition.

J. M. R.
W. W. J.

August, 1881.

CONTENTS.

CHAPTER I.

FUNCTIONS, RATES, AND DERIVATIVES.

I.

II.

III.

CHAPTER II.

THE DIFFERENTIATION OF ALGEBRAIC FUNCTIONS.

IV.

CHAPTER III.

The Differentiation of Transcendental Functions.

CHAPTER IV.

SUCCESSIVE DIFFERENTIATION.

CHAPTER V.

THE EVALUATION OF INDETERMINATE FORMS.

CHAPTER VIII.

Curve Tracing.

XXVI.

XXVII.

XXVIII.

XXIX.

XXX.

CHAPTER IX.

Equations and Constructions of Certain Higher Plane Curves.

XXXI.

CHAPTER X.

Applications of the Differential Calculus to Plane Curves.

XXXII.

XXXIII.

XXXIV.

XXXIX.

XL.

XLI.

THE

DIFFERENTIAL CALCULUS.

CHAPTER I.

FUNCTIONS, RATES, AND DERIVATIVES.

I.

Functions.

1. A QUANTITY which depends for its value upon another quantity is said to be a *function* of the latter quantity. Thus x^2, $\tan x$, $\log(a + x)$, and a^x are functions of x.

The quantity upon which the function depends must be regarded as variable, and be represented in the analytical expression for the function by an algebraic symbol. This quantity is called the *independent variable*. It is essential that variation of the independent variable should actually produce variation of the function. Thus the quantities x^0, $x^2 + (a + x)(a - x)$, and $(\tan x + \cot x)\sin 2x$ are *not* functions of x, since each admits of expression in a form which does not involve x.

2. The notation $f(x)$ is employed to denote any function of x, and, when several functions of x occur in the same in-

vestigation, such expressions as $F(x)$, $F'(x)$, $\phi(x)$, etc., are employed, the enclosed letter always denoting the independent variable. When expressions like $f(1)$, $f(a)$, $f(2x)$, or $f(0)$ are employed, it must be understood that the enclosed quantity is to be substituted for x in the expression which defines $f(x)$. Thus, if we have

$$f(x) = x^2 + x,$$

$$f(1) = 2, \qquad f(2x) = 4x^2 + 2x, \qquad \text{and} \qquad f(0) = 0.$$

Again, if $$F(x) = \log_a x \qquad (a > 1).$$

$$F(1) = 0, \qquad F(0) = -\infty, \qquad \text{and} \qquad F(a) = 1.$$

3. When x denotes the independent variable upon which a function depends, any quantity independent of x is, in contradistinction, called a *constant;* both when it is an absolute constant, like 1, $\sqrt{2}$, or π, and when it is denoted by a symbol, like a, u, or y, to which any value can be assigned. Thus, when a^x is denoted by $f(x)$, it is considered simply as a function of x, and a is regarded as a constant.

When it is desired to express that a quantity is a function of two quantities, both the symbols denoting them are placed between marks of parenthesis. Thus, since a^x is a function of x and a, we may write

$$f(x, a) = a^x.$$

Accordingly we have

$$f(y, b) = b^y, \qquad f(3, 2) = 8, \qquad \text{and} \qquad f(2, 3) = 9.$$

4. It is often convenient to represent the value of a function of x by a single letter; thus, for example, $y = x^2$. When this notation is used, if we represent the independent variable x by the abscissa of a point, and the function y by the corre-

sponding ordinate, a curve may be constructed which will graphically represent the function, and will serve to illustrate its peculiarities.

Rectangular coordinates are usually employed for this purpose. See diagram, Art. 10.

A function of the form

$$y = mx + b,$$

m and b being constants, is represented by a straight line. Functions of this form are, for this reason, called *linear functions*.

Implicit Functions.

5. When an equation is given involving two variables x and y, either variable is obviously a function of the other; and the former variable, when its value is not *directly* expressed in terms of the other, is said to be an *implicit* function of the latter. Thus, if we have

$$ax^2 - 3axy + y^3 - a^3 = 0,$$

either variable is an implicit function of the other.

By solving the above equation for x, we obtain

$$x = \frac{3y}{2} \pm \sqrt{\left(a^2 + \frac{9y^2}{4} - \frac{y^3}{a}\right)}.$$

In this form of the equation, x is said to be an *explicit* function of y.

This example will serve to illustrate the fact, that from a single equation involving two variables, there may be derived two or more explicit functions of the same variable. In the above case, x is said to be a *two-valued* function of y; while, since the equation is of the third degree in y, the latter is a *three-valued* function of x.

Inverse Functions.

6. If $y = f(x)$, x is some function of y; we may therefore write

$$y = f(x), \qquad \text{whence} \qquad x = \phi(y).$$

Each of the functions f and ϕ is then said to be the *inverse function* of the other. Thus, if

$$y = a^x, \qquad \text{we have} \qquad x = \log_a y;$$

hence each of these functions is the inverse of the other. So also the square and the square root are inverse functions.

7. In the case of the trigonometric functions, a peculiar notation for the inverse functions has been adopted. Thus, if we have.

$$x = \sin\theta, \qquad \text{we write} \qquad \theta = \sin^{-1}x.$$

Whenever trigonometric functions are employed in the Calculus, the symbol representing the angle always denotes the *circular measure* of the angle; that is, the ratio of the arc to the radius. Hence $\sin^{-1}x$ may be read either "the inverse sine of x," or "the arc whose sine is x."

The inverse trigonometric functions are evidently many-valued. See Art. 54.

The Classification of Functions.

8. With reference to its *form*, an explicit function is either *algebraic* or *transcendental.*

An *algebraic function* is expressed by a definite combination of algebraic symbols, in which the exponents do not involve the independent variable.

All functions not algebraic are classed as *transcendental.* Under this head are included *exponential* functions; that is, those in which one or more exponents are functions of the variable, as, for example, a^x, $x a^{\sqrt{x}}$, etc.: logarithmic functions: the direct and inverse trigonometric functions, and other forms which arise in the higher branches of mathematics.

9. With reference to its mode of variation, a function is said to be an *increasing function* when it increases and decreases with x; and a *decreasing function* when it decreases as x increases, and increases as x decreases. Thus, it is evident that x^3 is always an increasing function of x, while $\frac{1}{x}$ is always a decreasing function of x. Again, $\tan x$ is always an increasing function, but $\sin x$ is sometimes an increasing and sometimes a decreasing function of x.

10. The increase and decrease here considered are *algebraic.* For example, x^2 is an increasing function when x is positive, but when x is negative it becomes a decreasing function; for, when x is negative and algebraically increasing, x^2 is decreasing.

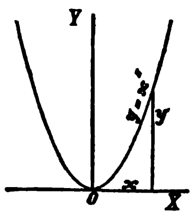


FIG. 1.

The curve $y = x^2$ which illustrates this function is constructed in Fig. 1. Since algebraic increase in the value of x is represented by motion from left to right, whether the moving point is on the left or on the right of the axis of y, the downward slope of the curve on the left of the origin indicates that x^2 is a decreasing function when x is negative.

Expressions involving an Unknown Function.

11. An expression involving $f(x)$, as, for example, $x f(x)$ or $F[f(x)]$, is generally a function of x; but it may happen

that such an expression has a value independent of x. Thus, suppose that, in the course of an investigation, the following equation presents itself :—

$$xf(x) = zf(z),$$

in which f denotes an unknown function, and x and z are entirely independent arbitrary quantities. When this is the case, we can make z a fixed quantity, and give to x any value whatever; that is, we can make x a variable and z a constant; but if z is a constant, $zf(z)$ is likewise a constant, we can, therefore, write

$$xf(x) = c,$$

c being an unknown constant. Hence we have

$$f(x) = \frac{c}{x}.$$

The value of the constant c is readily found, if we know the value of $f(x)$ corresponding to any one value of x.

Examples I.

✓ 1. (α) For what value of n does x^n cease to be a function of x? (β) For what values of x does it cease to be a function of n?

(α) When $n = 0$. (β) When $x = 1$, or $x = 0$.

✓ 2. If $y\left(1 - \dfrac{a-x}{a+x}\right) = x + \dfrac{ax - x^2}{a+x}$, show that y is a function of a, but not of x. $2x \cdot y = 2ax$ ∴ $y = a$

✓ 3. Show that $\sin x \tan \frac{1}{2}x + \cos x$ is not a function of x. $\tan\frac{x}{2} = \frac{\sin x}{1+\cos x}$ $= 1$

✓ 4. If $y = x + \sqrt{(1 + x^2)}$, show that $\underbrace{y^2 - 2xy}_{=1}$ is not a function of x.

✓ 5. If $f(x) = x^2$, find the value of $f(x + h)$; of $f(2x)$; of $f(x^2)$; of $f(x^2 - x)$; of $f(1)$; $f(12)$; $f[f(x)]$.

$$f(x + h) = x^2 + 2hx + h^2.$$

$f(2x) = 4x^2$
$f(x^2) = x^4$
$f(x^2-x) = (x^2-x)^2 = x^2(x-1)^2$
$f(1) = 1$ $f[f(x)] = (x^2)^2 = x^4$
$f(12) = 144$

6. If $f(\theta) = \cos\theta$, find the value of $f(0)$; of $f(\frac{1}{4}\pi)$; of $f(\frac{1}{2}\pi)$; of $f(\pi)$.
1 $\quad \frac{1}{2}\sqrt{2}$ $\quad 0$ $\quad -1$

7. If $F(x) = a^x$, give the value of $F(a)$; of $F(1)$; of $F(0)$. Also show that in this case $[F(x)]^2 = F(2x)$.
a^a $\quad a$ $\quad 1$ $\quad a^{2x} = a^{2x}$

8. Given $y^2 - 2ay + x^2 = 0$, make y an explicit function of x.

$$y = a \pm \sqrt{(a^2 - x^2)}.$$

9. Given $1 + \log_a y = 2\log_a (x + a)$, make y an explicit function of x.

$$y = \frac{(x+a)^2}{a}.$$

10. Given the equations—

$$n + 1 = n(\cos^2\theta' + \cos\theta' \cos\theta + \cos^2\theta),$$

and

$$n - 1 = n(\sin^2\theta' + \sin\theta' \sin\theta + \sin^2\theta);$$

eliminate n, and make θ' an explicit function of θ. Also make n an explicit function of θ.

$$\theta' = \theta \pm \tfrac{1}{2}\pi, \text{ and } n = \mp \frac{1}{\sin\theta \cos\theta}.$$

11. Given $\sin^{-1} x + \sin^{-1} y = \alpha$, make y an explicit function of x.
$\alpha_1 + \alpha_2 = \alpha$
$y = \sin\alpha_2 = \sin(\alpha - \alpha_1) =$

$$y = \sin\alpha\sqrt{(1 - x^2)} - x\cos\alpha.$$

12. Given $\tan^{-1} x + \tan^{-1} y = \alpha$, make y an explicit function of x.

$$y = \frac{\tan\alpha - x}{1 + x\tan\alpha}.$$

13. Given $xy - 2x + y = n$, show that y is not a function of x when $n = 2$.
$(1+x)(y-2) = 0, \quad y = 2$

14. If $y = \dfrac{2x - 1}{3x - 2}$, show that the inverse function is of the same form.

15. If $y = f(x) = \dfrac{1 + x}{1 - x}$, find $z = f(y)$, and express z as a function of x.

$$z = -\frac{1}{x}.$$

16. If both f and ϕ denote increasing functions, or, if both denote decreasing functions, show that $\phi[f(x)]$ is an increasing function. Also show that the inverse of an increasing function is an increasing function.

17. Find the inverse of the function, $y = \log_\epsilon [x + \sqrt{(1 + x^2)}]$.

$$x = \tfrac{1}{2}(\epsilon^y - \epsilon^{-y}).$$

18. If $f(x)$ be an unknown function having the property

$$f(x) + f(y) = f(xy),$$

prove that $\qquad f(1) = 0.$

Put $y = 1$.

19. If $f(x)$ has the property

$$f(x+y) = f(x) + f(y),$$

prove that $f(0) = 0$. Also prove that the function has the property

$$f(px) = pf(x),$$

in which p is a positive or negative integer.

For positive integers, put $y = x$, $2x$, $3x$, *etc., in the given equation; for negative integers, put* $y = -x$.

20. If f denotes the same function as in Example 19, prove that

$$f(mx) = mf(x),$$

m denoting any fraction.

Solution:—

Putting $z = \frac{p}{q}x$, $\qquad qz = px,$

$$f(qz) = f(px);$$

hence, by Example 19, $\qquad qf(z) = pf(x),$

or $\qquad f(z) = \frac{p}{q}f(x),$

$\therefore \qquad f\left(\frac{p}{q}x\right) = \frac{p}{q}f(x).$

21. Given, the property of the same function proved in Example 20; viz.,

$$f(mx) = mf(x);$$

by putting z for mx, show that

$$\frac{1}{z}f(z) = \frac{1}{x}f(x),$$

and thence deduce the form of the function. *See Art.* 11.

$$f(x) = cx.$$

22. Given, $[\phi(x)]^z = [\phi(z)]^x$, and $\phi(1) = e$, determine $\phi(x)$.

$$\phi(x) = e^{\frac{1}{x}}$$

23. Given $\phi(x) + \phi(y) = \phi(xy)$

prove $\phi(x^m) = m\,\phi(x)$,

and thence prove $\phi(x) = c \log x$.

Use the methods of Examples 19, 20, *and* 21.

II.

Rates.

12. In the Differential Calculus, variable quantities are regarded as undergoing continuous variation in magnitude, and the *rates* of variation, denoted by appropriate symbols, are employed in connection with the values of the variables themselves.

If a varying quantity be represented by the distance of a point moving in a straight line from a fixed origin taken on that line, the velocity of the moving point will represent the rate of increase or decrease of the varying quantity.

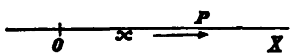


FIG. 2.

Thus O (Fig. 2) being the fixed origin and OP a variable denoted by x, P is the moving point whose velocity represents the rate of x. The velocity of P, or the rate of x, is regarded as positive when P moves in the direction in which x *increases algebraically;* thus, taking the direction OX, or toward the right, as the positive direction in laying off x, the

velocity is positive when P moves toward the right, whether its position be on the right or on the left of the origin. Accordingly, a rate of *algebraic decrease* is considered as negative, and would be represented by a point moving toward the left.

Constant Rates.

13. The rate of a quantity like the velocity of a point may be either constant or variable. A velocity is uniform or constant, when the spaces passed over in any equal intervals of time are equal, or, in other words, *when the spaces passed over in any intervals of time are proportional to the intervals.*

The numerical measure of a uniform velocity *is the space passed over in a unit of time;* then if t denote the time elapsed from an assumed origin of time, and k the space passed over by a moving point in a unit of time, kt will denote the space passed over in the time t. Hence, whenever the velocity is uniform, the quotient obtained by dividing the number of units of space by the number of units of time occupied in describing this space is constant, and serves as the numerical measure of the velocity.

14. Now, if x be a quantity having a uniform rate k, it will be represented by the distance from the origin of a point having the uniform velocity k, and if a denote the value of x when t is zero, we shall have

$$x = a + kt. \qquad \qquad (1)$$

This formula expresses a uniformly varying quantity as a function of t. When x is a uniformly decreasing quantity, k is, of course, negative.

Conversely, if x, when expressed as a function of t, is of the form (1), involving the first power only of t, then x is a quantity having a uniform rate, and the coefficient k is a measure of this rate.

Variable Velocities.

15. If the velocity of a point be *not* uniform, *its numerical measure at any instant is the number of units of space which would be described in a unit of time, were the velocity to remain constant from and after the given instant.*

Thus, when we speak of a body as having at a given instant a velocity of 32 feet per second, we mean that should the body continue to move during the whole of the next second, with the same velocity which it had at the given instant, 32 feet would be described. The *actual* space described may be greater or less, in consequence of the change in velocity which takes place during the second; it is, for instance, greater than the measure of the velocity at the beginning of the second, in the case of a falling body, because the velocity increases throughout the second.

16. Attwood's machine for determining experimentally the velocities acquired by falling bodies furnishes a familiar example of the practical application of the principle embodied in the above definition.

This apparatus consists essentially of a thread passing over a fixed pulley, and sustaining equal weights at each extremity, the pulley being so constructed as to offer but slight resistance to turning. On one of the weights a small bar of metal is placed, which, destroying the equilibrium, causes the weight to descend with an increasing velocity. To determine the value of this velocity at any point, a ring is so placed as to intercept the bar at that point, and allow the weight to pass. Thus, the sole cause of the variation of the velocity having been removed, the weight moves on uniformly with the required velocity, and the space described during the next second becomes the measure of this velocity.

Variable Rates.

17. When x is a function of t, but not of the form expressed by equation (1), Art. 14—that is, when the function is not linear—the rate of x will be variable. To obtain the measure of this rate at any given instant, we employ the same principle as in the case of a variable velocity. Thus, let x be represented by OP as in Fig. 2, Art. 12, let the symbol dt denote an assumed interval of time, and let dx denote the space which would be described in the time dt, were P to move with the velocity which it has at the given instant unchanged throughout the interval of time dt. Then the space which would be described in a unit of time is, evidently,

$$\frac{dx}{dt},$$

which is therefore the measure of the velocity of P, or the rate of x.

This ratio is in general variable, but, when x is of the form $a + kt$, it has been shown in Art. 14 that k is the measure of the rate; we therefore have

$$\frac{dx}{dt} = k, \qquad \text{when} \qquad x = a + kt.$$

Differentials.

18. The quantities dx and dt are called respectively "the differential of x" and "the differential of t."

In accordance with the definition of dx given in the preceding article, the *differential* of a variable quantity at any instant is the increment which would be received in the time dt, were the quantity to continue to increase uniformly during that interval of time with the rate it has at the given

instant. *The quotient obtained by dividing the differential of any quantity by* dt *is therefore the measure of the rate of the quantity.*

The differential of a quantity is denoted by prefixing d to the symbol denoting the quantity; when the symbol denoting the quantity is not a single letter it is usually enclosed by marks of parenthesis to avoid ambiguity. Thus, $d(x^2)$, $d(xy)$, $d(\tan x)$, $d(a^x + x^n)$, etc.

The Differentials of Polynomials.

19. Let x and y denote two variable quantities, and let a and b denote particular simultaneous values of x and y, while k and k' denote corresponding values of the rates of x and y.

Now, if x and y should continue to vary with these rates, their values would (see Art. 14) be expressed by

$$x = a + kt,$$

and $$y = b + k't,$$

whence $$x + y = a + b + (k + k')\,t.$$

Thus the quantity $x+y$ would become a uniformly varying quantity, and, by Art. 14, its rate would be $k+k'$, which, therefore, is the measure of the rate of $x+y$ at the instant when x and y have the rates k and k'. Consequently,

$$\frac{d(x+y)}{dt} = k + k' = \frac{dx}{dt} + \frac{dy}{dt}.$$

Now, since k and k' denote any values of the rates, this equation is universally true. We have, therefore,

$$d(x+y) = dx + dy. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This formula is easily extended to the sum of any number of variables. Thus,

$$d(x+y+z+\cdots) = dx + d(y+z+\cdots) = dx + dy + dz + \ldots \quad . \quad (2)$$

20. The differential of a constant is evidently zero, hence

$$d(x+h) = dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Again, if $\qquad y = -x, \qquad\qquad y + x = 0,$

hence, by equation (1), since zero is a constant, we have

$$dy + dx = 0, \qquad \text{or} \qquad dy = -dx;$$

that is, $\qquad d(-x) = -dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$

The differential of a negative term is therefore the negative of the differential of the term taken positively.

It appears, on combining the results expressed in equations (2), (3), and (4), that *the differential of a polynomial is the algebraic sum of the differentials of its terms; and that constant terms disappear from the result.*

The Differential of a Term having a Constant Coefficient.

21. Let the term be denoted by mx, m denoting a constant.

Resuming equation (2), Art. 19; viz.,

$$d(x+y+z+\cdots) = dx + dy + dz + \cdots,$$

and denoting the number of terms by p, we put

$$x = y = z = \cdots,$$

thus obtaining $\qquad d(px) = p\,dx, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$

p denoting an integer.

To extend equation (1) to the case in which m denotes a fraction, let

$$z = \frac{p}{q}x, \qquad \text{then} \qquad qz = px.$$

By applying equation (1) we obtain

$$q\,dz = p\,dx, \qquad \text{or} \qquad dz = \frac{p}{q}\,dx;$$

that is, $$d\left(\frac{p}{q}x\right) = \frac{p}{q}\,dx.$$

Hence generally, when m is positive,

$$d(mx) = m\,dx. \quad \ldots \ldots \quad (2)$$

Since $d(-x) = -dx$, this equation is true likewise when m is negative.

It therefore follows that *the differential of a term having a constant coefficient is equal to the product of the differential of the variable factor by the constant coefficient.*

Examples II.

1. Find the differential of $\frac{2x}{3a}$, and of $\frac{x}{m-2}$.
$\frac{2dx}{3a}$, and $\frac{dx}{m-2}$. ✓

2. Find the differential of $\frac{x-a}{m^2}$, and of $\frac{a-x}{m^2}$.
$\frac{dx}{m^2}$, and $-\frac{dx}{m^2}$. ✓

3. Find the differential of $\frac{a+b+(a-b)x}{a^2-b^2}$.
$\frac{dx}{a+b}$. ✓

4. Find the differential of $\frac{a+x}{a+b}$, and of $\frac{b(x+y)}{a(a+b)}$.
$\frac{dx}{a+b}$, and $\frac{b(dx+dy)}{a(a+b)}$. ✓
✓

5. Given $ay + bx + 2cx + ab = 0$, to find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = -\frac{b + 2c}{a}.$$

6. Given $y \log a + x \sin \alpha - y \cos \alpha - ax + \tan \alpha = 0$, to find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{a - \sin \alpha}{\log a - \cos \alpha}.$$

7. Given $ay \cos^2 \alpha - 2b(1 - \sin \alpha)x = b(a - x \cos^2 \alpha)$, to find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{b(1 - \sin \alpha)}{a(1 + \sin \alpha)}.$$

8. Given $a^2 + 2(1 + \cos \alpha)y = (x + y) \sin^2 \alpha$, to find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \tan^2 \frac{\alpha}{2}.$$

9. Given $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, to express dz in terms of dx and dy.

$$dz = -\frac{c}{a}\,dx - \frac{c}{b}\,dy.$$

√ 10. A man whose height is 6 feet walks directly away from a lamp-post at the rate of 3 miles an hour. At what rate is the extremity of his shadow travelling, supposing the light to be 10 feet above the level pavement on which he is walking?

Draw a figure, and denote the variable distance of the man from the lamp-post by x, and the distance of the extremity of his shadow from the post by y. $x = a + Kt = Kt$

$\frac{y}{x} = \frac{10}{4}$, $\frac{dy}{dt} = \frac{5}{2}K = \frac{5}{2} \times 3 = 7\frac{1}{2}$

$7\frac{1}{2}$ miles per hour.

11. At what rate does the man's shadow (Ex. 10) increase in length?

III.

Differentials of Functions of an Independent Variable.

22. When the variables involved in any mathematical investigation are functions of an independent variable x, the latter may be assumed to have a rate denoted by $\frac{dx}{dt}$, in which

dx is arbitrary. So also the corresponding rate of y will be denoted by $\frac{dy}{dt}$, and, if y is a function of x, the value of dy will depend in part upon the assumed value of dx.

To differentiate a function of x is to express its differential in terms of x and dx.

It is to be understood, of course, that the differentials involved in an equation are all taken with reference to the same value of dt.

If two quantities are always equal, their simultaneous rates are evidently equal; and hence their differentials are likewise equal. We can therefore *differentiate an equation;* that is, express the equality of the differentials of its members; provided the equation is true for all values of the variables involved. Thus, from the identical equation

$$(x+h)^2 = x^2 + 2hx + h^2,$$

it follows that $\quad d[(x+h)^2] = d(x^2) + 2h\,dx.$

The Derivative.

23. Before proceeding to the differentiation of the various functions of x, it is necessary to show that, if

$$y = f(x), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

the ratio

$$\frac{dy}{dx}$$

has a definite value for each value of x, *independent of the assumed value of* dx.

Let a particular value of x be denoted by a, and let the corresponding value of dx be an arbitrary quantity.

Now, although dx is arbitrary, since dt is likewise arbitrary, the rate of x, that is, the ratio

$$\frac{dx}{dt}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

may be assumed to have a certain *fixed value* at the instant when $x = a$. The corresponding value of the rate of y, denoted by

$$\frac{dy}{dt}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

evidently depends solely upon the rate of x and upon the form of the function f in equation (1). Hence, when the value of the rate (2) is fixed, the value of (3) is also definitely fixed.

Denoting these fixed values by k and k', we have, when $x = a$,

$$\frac{dx}{dt} = k, \text{ and } \frac{dy}{dt} = k', \text{ whence } \frac{dy}{dx} = \frac{k'}{k}.$$

Hence, corresponding to a particular value a of x, there exists a determinate value $\frac{k'}{k}$, of the ratio $\frac{dy}{dx}$, notwithstanding the fact that dx has an arbitrary value; in other words, *the value of the ratio* $\frac{dy}{dx}$ *is independent of the arbitrary value of* dx.

24. It is obvious that, in general, this ratio will have different values corresponding to different values of x, and hence that it may be expressed as a function of x, and denoted by $f'(x)$; thus,—

$$\frac{dy}{dx} = f'(x). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The form of this new function f' will evidently depend upon that of the given function f.

The function $f'(x)$ is called the *derivative* of $f(x)$, and, since equation (1) may be written in the form

$$dy = f'(x)\, dx,$$

it is also called the *differential coefficient* of y regarded as a function of x.

When, however, the given function $f(x)$ is of the linear form

$$y = mx + b,$$

the derivative is no longer a function of x, but is a constant, since the value of y gives

$$dy = m\,dx,$$

or
$$\frac{dy}{dx} = m.$$

The Geometrical Meaning of the Derivative.

25. Representing the corresponding values of x and y by the rectangular coordinates of a moving point, if this point move in a uniform direction, so as to describe a straight line,—that is, if y be a linear function of x,—the value of $\frac{dy}{dx}$ will be constant, by the preceding article. Hence, in the general case, when this ratio is variable, the point will move in a variable direction.

If we denote the inclination of this direction to the axis of x by ϕ, the value of ϕ will vary with the value of x, and the point will describe a curve.

The tangent line to a curve is defined as follows:—

*The tangent to a curve at any point is the straight line which passes through the point, and has the direction of the curve at that point.**

Hence, for any point of the curve, ϕ denotes the inclination to the axis of x of the tangent line at that point.

---

* It will be shown hereafter (Art. 49) that, in the case of the circle, this general definition of a tangent line agrees with that usually given in Plane Geometry.

26. Now, if a point, at first moving in the curve, should, after passing the point whose abscissa is a, so move that the rates $\frac{dx}{dt}$ and $\frac{dy}{dt}$ retain the values which they had at the instant of passing the given point, the direction of its motion will become constant, and the point will describe a straight line tangent to the curve at the given point.

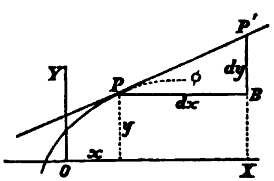


Fig. 3.

The value of dx may be represented by an arbitrary increment of x as in Fig. 3; the value of dy will then be represented by the corresponding increment which would be received by y, were the point moving in the tangent line, as indicated in the diagram. Hence

$$\frac{dy}{dx} = \tan\phi,$$

which is evidently independent of the assumed value of dx.* It follows that the value of the derivative of $f(x)$, for any value of x, is represented by the trigonometric tangent of the inclination to the axis of x of the curve $y = f(x)$, at the point corresponding to the given value of x.

27. The moving point, which is conceived to describe the curve, may pass over it in either of two directions differing by 180°. The two corresponding values of ϕ give, however, the same value of $\tan\phi$, since $\tan(\phi \pm 180°) = \tan\phi$.

Thus, in Fig. 3, the point P may be regarded as moving so as to increase x and y, in which case both dx and dy will be positive, and ϕ will be in the first quadrant; or P may

* In other words, the value of the derivative is determined by the form of the function f which determines the curve, and the value of x which fixes the position of P.

move in the opposite direction, making dx and dy negative, and placing ϕ in the third quadrant. In either case, $\frac{dy}{dx}$ or $\tan\phi$ is positive.

28. It is evident that when $f(x)$ is an increasing function, as in Fig. 3, $\frac{dy}{dx}$ is positive, and that when it is a decreasing function, $\frac{dy}{dx}$ is negative.

Thus the sign of $f'(x)$ for any value of x is positive or negative according as $f(x)$ is, for that value of x, an increasing or a decreasing function. For example, it is evident that the value of the derivative of $\sin x$ must be positive when x is between 0 and $\frac{1}{2}\pi$, negative when x is between $\frac{1}{2}\pi$ and $\frac{3}{2}\pi$, and so on.

When the notation $\frac{dy}{dx}$ is used, the value of the derivative corresponding to a particular value a of x is expressed by $\left.\frac{dy}{dx}\right]_a$ which is equivalent to $f'(a)$. See Art. 2.

Examples III.

1. If a point move in the straight line $2y - 7x - 5 = 0$, so that its ordinate decreases at the rate of 3 units per second, at what rate is the point moving in the direction of the axis of x?

$$\frac{dx}{dt} = -\frac{6}{7}.$$

2. If a point starting from $(0, b)$ move so that the rates of its coordinates are k and k', show that its path is $y = mx + b$, m being equal to $\frac{k'}{k}$.

Express x and y in terms of t (Art. 14), and eliminate t.

3. If a point moving in a curve passes through the point (5, 3)

moving at equal rates upward and toward the left, find the value of $\left.\frac{dy}{dx}\right]_3$, also the equation of the tangent line to the curve at the given point.

$$\left.\frac{dy}{dx}\right]_3 = -1, \text{ and } y+x=8.$$

4. If a point is moving in the straight line

$$x \cos \alpha + y \sin \alpha = p,$$

its rate in the positive direction of the axis of x being $l \sin \alpha$, what is its rate of motion in the direction of the axis of y?

$$-l \cos \alpha.$$

5. Given $ay \sin \alpha - ax + ax \cos \alpha - b^2 \sec \alpha = 0$; show that ϕ is constant and equal to $\frac{1}{2}\alpha$.

6. If $f(x) = \tan x$, show that $f'(x)$ must always be positive.

7. Show, by tracing the curve, that if $y = x^3$, $\frac{dy}{dx}$ can never be negative.

CHAPTER II.

THE DIFFERENTIATION OF ALGEBRAIC FUNCTIONS.

IV.

The Square.

29. IN establishing the formulas for the differentiation of the simple algebraic functions of an independent variable, we find it convenient to begin with the square. The object of this article is, therefore, to express $d(x^2)$ in terms of x and dx.

We first deduce a relation between two values of the derivative of the function and the corresponding values of the independent variable; for this purpose, we assume two values of the variable having a constant ratio m. Thus, if

$$z = mx, \qquad\qquad z^2 = m^2 x^2.$$

Differentiating by equation (2), Art. 21,

$$dz = m\,dx, \qquad \text{and} \qquad d(z^2) = m^2\, d(x^2);$$

dividing, we obtain

$$\frac{d(z^2)}{dz} = m\,\frac{d(x^2)}{dx}.$$

Whence, dividing by $z = mx$ to eliminate m, we have

$$\frac{1}{z}\cdot\frac{d(z^2)}{dz} = \frac{1}{x}\cdot\frac{d(x^2)}{dx}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The derivatives $\frac{d(z^2)}{dz}$ and $\frac{d(x^2)}{dx}$ are, by Art. 23, functions of z and of x respectively, independent of the values of dz and dx; moreover, equation (1) is true for all values of x and z, these quantities being entirely independent of each other, since the arbitrary ratio m has been eliminated. Therefore, either of these quantities may be assumed to have a fixed value, while the other is variable; hence it follows that the value of each member of this equation must be a fixed quantity, independent of the value of x or of z. Denoting this fixed value by c, we therefore write

$$\frac{1}{x} \cdot \frac{d(x^2)}{dx} = c,$$

or

$$d(x^2) = c\,x\,dx \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

30. *To determine the unknown constant c,* we apply this result to the identity

$$(x+h)^2 = x^2 + 2hx + h^2.$$

Differentiating each member (Art. 22) by equation (2), we have

$$c\,(x+h)\,d(x+h) = c\,x\,dx + 2h\,dx;$$

since $d(x+h) = dx$, this equation reduces to

$$c\,h\,dx = 2h\,dx,$$

or

$$(c-2)\,h\,dx = 0.$$

Now, since h and dx are arbitrary quantities, this equation gives

$$c = 2;$$

this value of c substituted in (2) gives

$$d(x^2) = 2x\,dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

That is, *the differential of the square of a variable equals twice the product of the variable and its differential.*

31. Employing the derivative notation, this result may also be expressed thus :—

If $$f(x) = x^2, \qquad f'(x) = 2x.$$

This derivative is negative for negative values of x, therefore, for these values, x^2 is a decreasing function, as already mentioned (Art. 10) in connection with the curve illustrating this function.

Since x and dx are arbitrary, we may substitute for them any variable and its differential. Equation (a) therefore enables us to differentiate the square of any variable whose differential is known. Thus,—

$$d(5x - 3)^2 = 2(5x - 3)\,5dx = 10(5x - 3)\,dx.$$

Again, $$d(ax^2 + bx)^2 = 2(ax^2 + bx)\,d(ax^2 + bx)$$
$$= 2(ax^2 + bx)\,(2ax + b)\,dx.$$

The Square Root.

32. To derive the differential of the *square root*, we put

$$y = \sqrt{x},$$

whence $$y^2 = x;$$

differentiating by (a), $$2y\,dy = dx,$$

or $$dy = \frac{dx}{2y}.$$

But $y = \sqrt{x}$, $\therefore$ $$d(\sqrt{x}) = \frac{dx}{2\sqrt{x}}. \qquad (b)$$

That is, *the differential of the square root of a variable is equal to the quotient arising from dividing the differential of the variable by twice the given square root.*

Thus, $$d[\sqrt{(a^2 - x^2)}] = \frac{-x\,dx}{\sqrt{(a^2 - x^2)}},$$

or, using derivatives,

$$\frac{d[\sqrt{(a^2 - x^2)}]}{dx} = -\frac{x}{\sqrt{(a^2 - x^2)}}.$$

Examples IV.

1. Differentiate $(2x + 3)^2$, and find the numerical value of its rate, when x has the value 8, and is decreasing at the rate of 2 units per second.

The differential required is denoted by $d[(2x + 3)^2]$, *and the rate by* $\dfrac{d[(2x + 3)^2]}{dt}$; *the given rate* $\dfrac{dx}{dt} = -2$. $\dfrac{d(2x+3)^2}{dt} = \dfrac{d(2x+3)^2}{dx}\dfrac{dx}{dt} = 2(2x+3).2(-2)$

-152 units per second.

2. Find the numerical value of the rate of $(x^2 - 2x)^2$, when $x = 3$, and is increasing at the rate of $\frac{1}{3}$ of one unit per second.

Differentiate the given expression before substituting.

12 units per second.

3. Find the numerical value of the rate of $\sqrt{(y^2 + x^2)}$, when $y = 7$ and $x = -7$, if y is increasing at the rate of 12 units per second, and x at the rate of 4 units per second.

$4\sqrt{2}$ units per second.

4. If $f(x) = x - \sqrt{(x^2 - a^2)}$, find $f'(x)$, and show that $f(x)$ is a decreasing function.

$$f'(x) = 1 - \frac{x}{\sqrt{(x^2 - a^2)}}.$$

5. Differentiate the identity $(\sqrt{x} + \sqrt{a})^2 = x + a + 2\sqrt{ax}$, and show that the result is an identity.

6. Differentiate $\sqrt{\left(\dfrac{x^2 - 2ax}{a^2 - 2ab}\right)}$.

The constant factor $\dfrac{1}{\sqrt{(a^2 - 2ab)}}$ *should be separated from the variable factor before differentiation.*

$$\frac{1}{\sqrt{(a^2 - 2ab)}} \cdot \frac{x - a}{\sqrt{(x^2 - 2ax)}}\,dx.$$

7. If $f(x) = (1 + x^2)^{\frac{1}{2}}$, $\qquad f'(x) = \dfrac{x}{(1 + x^2)^{\frac{1}{2}}}.$

8. If $f(x) = \sqrt{(a^2 + 2b^2 x + c x^2)}$, $\qquad f'(x) = \dfrac{b^2 + c x}{\sqrt{(a^2 + 2b^2 x + c x^2)}}.$

9. If $f(x) = \sqrt{[x + \sqrt{(1 + x^2)}]}$, $\qquad f'(x) = \dfrac{\sqrt{[x + \sqrt{(1 + x^2)}]}}{2\sqrt{(1 + x^2)}}.$

10. If $f(x) = \dfrac{a^2}{x - \sqrt{(x^2 - a^2)}}$, $\qquad f'(x) = 1 + \dfrac{x}{\sqrt{(x^2 - a^2)}}.$

Rationalize the denominator before differentiating.

11. Given $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, express $\dfrac{dy}{dx}$ in terms of x, and give the values of $\left.\dfrac{dy}{dx}\right]_0$ and $\left.\dfrac{dy}{dx}\right]_a$. $\qquad \dfrac{dy}{dx} = \mp \dfrac{b}{a} \cdot \dfrac{x}{\sqrt{(a^2 - x^2)}}.$

12. Given $y^2 = 4ax$, express $\dfrac{dy}{dx}$ in terms of x, also in terms of y, and give the values of $\left.\dfrac{dy}{dx}\right]_a$ and $\left.\dfrac{dy}{dx}\right]_{4a}$. $\qquad \dfrac{dy}{dx} = \sqrt{\dfrac{a}{x}} = \dfrac{2a}{y}.$

13. A man is walking on a straight path at the rate of 5 ft. per second; how fast is he approaching a point 120 ft. from the path in a perpendicular, when he is 50 ft. from the foot of the perpendicular?

Solution:—

Let x denote the variable distance of the man from the foot of the perpendicular, so that $\dfrac{dx}{dt}$ may denote the known velocity of the man, and let a denote the length of the perpendicular (120 ft.); then the distance of the man from the point is $\sqrt{(a^2 + x^2)}$, of which the rate of change is denoted by

$$\frac{d[\sqrt{(a^2 + x^2)}]}{dt} = \frac{x}{\sqrt{(a^2 + x^2)}} \frac{dx}{dt}.$$

At the instant considered, $x = 50$ ft., while $a = 120$ ft., and $\dfrac{dx}{dt} = -5$ ft.

per second. By substituting these values, we obtain $-1\frac{11}{18}$. Hence his distance from the point is diminishing (that is, he is approaching it) at the rate of $1\frac{11}{18}$ ft. per second.

14. If the side of an equilateral triangle increase uniformly at the rate of 3 ft. per second, at what rate per second is the area increasing, when the side is 10 ft.? $15\sqrt{3}$ sq. ft.

15. A stone dropped into still water produces a series of continually enlarging concentric circles; it is required to find the rate per second at which the area of one of them is enlarging, when its diameter is 12 inches, supposing the wave to be then receding from the centre at the rate of 3 inches per second. 36π sq. inches.

16. If a circular disk of metal expand by heat so that the area A of each of its faces increases at the rate of 0.01 sq. ft. per second, at what rate per second is its diameter increasing? $\frac{1}{100\sqrt{(\pi A)}}$ ft.

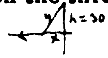
$A = \frac{1}{4}\pi D^2, \; D = \frac{2\sqrt{A}}{\sqrt{\pi}}$

$\frac{dA}{dt} = \frac{1}{2}\pi D \frac{dD}{dt}, \; \frac{dD}{dt} = \frac{.01 \times 2}{\pi D} = \frac{.01 \times 2\sqrt{\pi}}{\pi \cdot 2\sqrt{A}}$

17. A man standing on the edge of a wharf is hauling in a rope attached to a boat at the rate of 4 ft. per second. The man's hands being 9 ft. above the point of attachment of the rope, how fast is the boat approaching the wharf when she is at a distance of 12 ft. from it? 5 ft. per second.

18. A ladder 25 ft. long reclines against a wall; a man begins to pull the lower extremity, which is 7 ft. distant from the bottom of the wall, along the ground at the rate of 2 ft. per second; at what rate per second does the other extremity *begin* to descend along the face of the wall? 7 inches.

19. One end of a ball of thread is fastened to the top of a pole 35 ft. high; a man holding the ball 5 ft. above the ground moves uniformly from the bottom at the rate of five miles an hour, allowing the thread to unwind as he advances. What is the man's distance from the pole when the thread is unwinding at the rate of one mile per hour? $\frac{5}{4}\sqrt{6}$ ft.

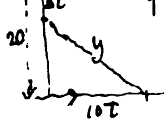


$y = \sqrt{x^2+h^2}, \; \frac{dy}{dt} = \frac{x}{\sqrt{x^2+h^2}}\frac{dx}{dt}$

$1 = \frac{x}{\sqrt{x^2+30^2}} \cdot 5$

20. A vessel sailing due south at the uniform rate of 8 miles per hour is 20 miles north of a vessel sailing due east at the rate of 10 miles an

(20)



$y^2 = (20-8t)^2 + (10t)^2$

$\frac{dy}{dt} = \frac{50t - 16(5-2t)}{\sqrt{4(5-2t)^2 + 25t^2}}$

hour. At what rate are they separating—(α) at the end of $1\frac{1}{2}$ hours? (β) at the end of $2\frac{1}{2}$ hours?

Express the distances in terms of the time. (α) $5\frac{1}{17}$ miles per hour.
(β) 10 " " "

21. When are the two ships mentioned in the preceding example neither receding from nor approaching each other?

Put the expression for their rate of separation equal to zero.

$50t = 11(5-2t)$ When $t =$ [illegible] of an hour.

22. Derive, by the method employed in Art. 29 to determine the differential of the square, the result $d\left(\frac{1}{x}\right) = \frac{c\,dx}{x^2}$, c being an unknown constant.

V.

The Product.

33. Let x and y denote any two variables; in order to derive the differential of their product, we express xy by means of squares, since we have already obtained a formula for the differentiation of the square. From the identity

$$(x+y)^2 = x^2 + 2xy + y^2,$$

we derive

$$xy = \tfrac{1}{2}(x+y)^2 - \tfrac{1}{2}x^2 - \tfrac{1}{2}y^2.$$

Differentiating, $\quad d(xy) = (x+y)(dx+dy) - x\,dx - y\,dy,$

therefore, $\quad d(xy) = y\,dx + x\,dy. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$

Since x and y denote any variables whatever, and dx and dy their differentials, we can substitute for x and y any variable expressions, and for dx and dy the corresponding differentials. Thus,

$$d[(1+x^2)\sqrt{(a^2-x^2)}] = \sqrt{(a^2-x^2)}\,2x\,dx - \frac{(1+x^2)x\,dx}{\sqrt{(a^2-x^2)}}$$
$$= \frac{2a^2 - 3x^2 - 1}{\sqrt{(a^2-x^2)}}\,x\,dx.$$

34. Formula (c) is readily extended to products consisting of any number of factors. Thus let $x_1 x_2 x_3 \ldots\ldots x_p$ denote the product of p variable factors, then

$$
\begin{aligned}
d(x_1 x_2 x_3 \cdots x_p) &= x_2 x_3 \cdots x_p\, dx_1 + x_1 d(x_2 x_3 \cdots x_p) \\
&= x_2 x_3 \cdots x_p\, dx_1 + x_1 x_3 \cdots x_p\, dx_2 + x_1 x_2 d(x_3 \cdots x_p) \\
&= x_2 x_3 \cdots x_p\, dx_1 + x_1 x_3 \cdots x_p\, dx_2 \cdots + x_1 x_2 \cdots x_{p-1} dx_p. \quad . \quad (c')
\end{aligned}
$$

The Reciprocal.

35. The differential of the reciprocal may now be obtained by means of the implicit form of this function.

Denoting the function by y, we have

$$y = \frac{1}{x} \quad \therefore \quad xy = 1.$$

Differentiating the latter equation by formula (c), we obtain

$$y\,dx + x\,dy = 0,$$

whence $$dy = -\frac{y\,dx}{x};$$

substituting the value of y,

$$d\left(\frac{1}{x}\right) = -\frac{dx}{x^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$$

Formula (d) enables us to differentiate any fraction of which the denominator alone is variable; thus,

$$d\left(\frac{a+b}{a+x}\right) = -(a+b)\frac{dx}{(a+x)^2}.$$

The Quotient.

36. By the term *quotient*, as used in this article, we mean a fraction whose numerator and denominator are both variable. In deriving its differential, the quotient is regarded as the product of its numerator by the reciprocal of its denominator. Thus, applying formulas (*c*) and (*d*),

$$d\left(\frac{x}{y}\right) = d\left(x\frac{1}{y}\right) = \frac{1}{y}\,dx + x\,d\left(\frac{1}{y}\right)$$

$$= \frac{dx}{y} - \frac{x\,dy}{y^2},$$

$$\therefore \qquad d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}. \quad \ldots \ldots \ldots \ldots \quad (e)$$

It will be noticed that the negative sign belongs to the term which contains the differential of the denominator.

As an illustration of the application of this formula, we have

$$d\left(\frac{2x-a}{x^2+b}\right) = \frac{2(x^2+b) - 2x(2x-a)}{(x^2+b)^2}\,dx = 2\,\frac{b+ax-x^2}{(x^2+b)^2}\,dx.$$

Formula (*e*) is to be used *only when both terms of the fraction are variable*; for, when the numerator is constant, the fraction is equivalent to the product of a constant and the reciprocal of a variable, and, when the denominator is constant, it is equivalent to the product of a constant by a variable factor. Thus, if it be required to differentiate the fraction $\frac{x^2+a^2}{ax}$, the use of formula (*e*) may be avoided by first making the transformation,

$$\frac{x^2+a^2}{ax} = \frac{x}{a} + \frac{a}{x};$$

since, in this form, one term of each fraction is constant. Hence,

$$d\left(\frac{x^2+a^2}{ax}\right)=\frac{dx}{a}-\frac{a\,dx}{x^2}.$$

The Power.

37. To obtain the differential of the power when the exponent is a positive integer, suppose each of the variables $x_1 x_2 x_3 \cdots x_p$ in formula (c'), Art. 34, to be replaced by x. The first member contains p factors, and the second p terms; the equation therefore reduces to

$$d(x^p)=px^{p-1}\,dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Next, when the exponent is a fraction, let

$$y=x^{\frac{p}{q}}, \qquad \text{then} \qquad y^q=x^p;$$

differentiating by (1), p and q being positive integers, we have

$$qy^{q-1}\,dy=px^{p-1}\,dx,$$

therefore,

$$dy=\frac{p}{q}\cdot\frac{x^{p-1}}{y^{q-1}}\,dx.$$

Substituting the value of y,

$$d(x^{\frac{p}{q}})=\frac{p}{q}\cdot\frac{x^{p-1}}{x^{p-\frac{p}{q}}}\,dx=\frac{p}{q}x^{\frac{p}{q}-1}\,dx. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Again, when the exponent is negative, we have

$$x^{-n}=\frac{1}{x^n}.$$

Differentiating by formula (d), Art. 35, we obtain

$$d(x^{-m}) = -\frac{d(x^m)}{x^{2m}},$$

and, since m is positive, we have, by (1) or (2),

$$d(x^{-m}) = -\frac{mx^{m-1}\,dx}{x^{2m}} = -m x^{-m-1}\,dx. \quad . \quad . \quad . \; (3)$$

Equations (1), (2), and (3) show that, for all values of n,

$$d(x^n) = n x^{n-1} dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \; (f)$$

By giving to n the values 2, $\frac{1}{2}$, and -1, successively, it is readily seen that this more general formula includes formulas (a), (b) and (d).

38. It is frequently advantageous to transform a given expression by the use of fractional or negative exponents, and employ formula (f) instead of formulas (b) and (d). Thus,

$$d\left[\frac{1}{(a^2 - 2x^2)^2}\right] = d\,(a^2 - 2x^2)^{-2} = 8\,(a^2 - 2x^2)^{-3}x\,dx,$$

and $$d\left[\frac{1}{\sqrt{(a+x)^3}}\right] = d(a+x)^{-\frac{3}{2}} = -\tfrac{3}{2}(a+x)^{-\frac{5}{2}}dx.$$

When the derivative of a function is required, it may be written at once instead of first writing the differential, since the former differs from the latter only in the omission of the factor dx, which must necessarily occur in every term. Thus, given

$$y = \frac{x}{\sqrt{(1+x^2)}} = x(1+x^2)^{-\frac{1}{2}},$$

we derive $$\frac{dy}{dx} = (1+x^2)^{-\frac{1}{2}} - \tfrac{1}{2}\,x\,(1+x^2)^{-\frac{3}{2}} \,.\, 2x = \frac{1}{(1+x^2)^{\frac{3}{2}}}.$$

Examples V.

1. From the identity $xy = \frac{1}{4}(x+y)^2 - \frac{1}{4}(x-y)^2$ derive the formula for differentiating the product.

2. Differentiate $\dfrac{a + bx + cx^2}{x}$.

Put the expression in the form $\dfrac{a}{x} + b + cx$. $\qquad \left(c - \dfrac{a}{x^2}\right)dx$.

3. Find the derivative of

$$y = \frac{a^2 - b^2}{a^2 - x^2}. \quad \textit{See remark, Art. } 35. \qquad \frac{dy}{dx} = (a^2 - b^2)\frac{2x}{(a^2 - x^2)^2}.$$

4. $y = \sqrt{(x^3 - a^3)}$. $\qquad \dfrac{dy}{dx} = \dfrac{3x^2}{2\sqrt{(x^3 - a^3)}}$.

5. $y = \dfrac{2x^4}{a^2 - x^2}$. = $\qquad \dfrac{dy}{dx} = \dfrac{4x^3(2a^2 - x^2)}{(a^2 - x^2)^2}$.

6. $y = (1 + 2x^2)(1 + 4x^3)$. $= 1+2x^2+4x^3+8x^5$ $\qquad \dfrac{dy}{dx} = 4x(1 + 3x + 10x^3)$.

7. $y = (a^2 + x^2)(b^2 + 3x^2)$. $\qquad \dfrac{dy}{dx} = 3(5x^3 + b^2x + 2a^2)x$.

8. $y = (1 + x)^4(1 + x^2)^2$. $\qquad \dfrac{dy}{dx} = 4(1 + x)^3(1 + x^2)(1 + x + 2x^2)$.

9. $y = (1 + x^m)^n + (1 + x^n)^m$.

$$\frac{dy}{dx} = mn[(1 + x^m)^{n-1}x^{m-1} + (1 + x^n)^{m-1}x^{n-1}].$$

10. $y = \dfrac{x^2 - 2a^2}{x - a}$. $= x + a - \dfrac{a^2}{x-a}$ $\qquad \dfrac{dy}{dx} = 1 + \dfrac{a^2}{(x - a)^2}$.

11. $y = \dfrac{a - x}{\sqrt{x}}$. $= ax^{-\frac{1}{2}} - x^{\frac{1}{2}}$ $\qquad \dfrac{dy}{dx} = -\dfrac{a + x}{2x^{\frac{3}{2}}}$.

12. $y = \dfrac{\sqrt{(x^2 - a^2)}}{x}$. $\qquad \dfrac{dy}{dx} = \dfrac{a^2}{x^2\sqrt{(x^2 - a^2)}}$.

13. $y = \dfrac{ab}{cx\sqrt{(x^2 - a^2)}}$. *See Art.* 38. $\qquad \dfrac{dy}{dx} = -\dfrac{ab}{c}\cdot\dfrac{2x^2 - a^2}{x^2(x^2 - a^2)^{\frac{3}{2}}}$.

14. $y = \dfrac{1}{\sqrt{(1 + x)}} + \dfrac{1}{\sqrt{(1 - x)}}$. $\qquad \dfrac{dy}{dx} = \tfrac{1}{2}[(1 - x)^{-\frac{3}{2}} - (1 + x)^{-\frac{3}{2}}]$.

15. $y = (1 + x)\sqrt{(1 - x)}$. $\qquad \dfrac{dy}{dx} = \dfrac{1 - 3x}{2\sqrt{(1 - x)}}$.

16. $y = (a + x)^3(b - x)^4x^2$.

$$\frac{dy}{dx} = x(a + x)^2(b - x)^3[2ab + (5b - 6a)x - 9x^2].$$

17. $y = \dfrac{x^n + 1}{x^n - 1}$. $\qquad \dfrac{dy}{dx} = -\dfrac{2nx^{n-1}}{(x^n - 1)^2}$.

18. $y = (3b + 2ax)^{\frac{3}{2}}(b - ax)$. $\qquad \dfrac{dy}{dx} = -5a^2x\sqrt{(3b + 2ax)}$.

19. $y = \dfrac{a^2 - b^2}{(2ax - x^2)^{\frac{3}{2}}}$.

Put in the form $(a^2 - b^2)(2ax - x^2)^{-\frac{3}{2}}$. $\qquad \dfrac{dy}{dx} = 3(a^2 - b^2)\dfrac{x - a}{(2ax - x^2)^{\frac{5}{2}}}$.

20. $y = \dfrac{x}{\sqrt{(a^2 - x^2)}}$. $\qquad \dfrac{dy}{dx} = \dfrac{a^2}{(a^2 - x^2)^{\frac{3}{2}}}$.

21. $y = \dfrac{bx}{\sqrt{(2ax - x^2)}}$. $\qquad \dfrac{dy}{dx} = \dfrac{abx}{(2ax - x^2)^{\frac{3}{2}}}$.

22. $y = \sqrt{\dfrac{1 + x}{1 - x}}$. $\qquad \dfrac{dy}{dx} = \dfrac{1}{(1 - x)\sqrt{(1 - x^2)}}$.

23. $y = \dfrac{x}{\sqrt{(a^2 + x^2)} - x}$.

Rationalize the denominator. $\qquad \dfrac{dy}{dx} = \dfrac{1}{a^2}\left[\dfrac{a^2 + 2x^2}{\sqrt{(a^2 + x^2)}} + 2x\right]$.

24. $y = \dfrac{x^3}{\sqrt{(1-x^3)}}$. $\qquad \dfrac{dy}{dx} = \dfrac{3x^2}{2} \cdot \dfrac{2-x^3}{(1-x^3)^{\frac{3}{2}}}$.

25. $y = \dfrac{x^3}{(1-x^2)^{\frac{3}{2}}}$. $\qquad \dfrac{dy}{dx} = \dfrac{3x^2}{(1-x^2)^{\frac{5}{2}}}$.

26. $y = \dfrac{x^n}{(1+x)^n}$. $\qquad \dfrac{dy}{dx} = \dfrac{n\,x^{n-1}}{(1+x)^{n+1}}$.

27. $y = \dfrac{1}{(a+x)^m (b+x)^n}$. $\qquad \dfrac{dy}{dx} = -\dfrac{na + mb + (m+n)x}{(a+x)^{m+1}(b+x)^{n+1}}$.

28. $y = \dfrac{2x^2-1}{x\sqrt{(1+x^2)}}$. $\qquad \dfrac{dy}{dx} = \dfrac{1+4x^2}{x^2(1+x^2)^{\frac{3}{2}}}$.

29. $y = \dfrac{\sqrt{(1+x^2)} + \sqrt{(1-x^2)}}{x}$. $\qquad \dfrac{dy}{dx} = -\dfrac{\sqrt{(1+x^2)} + \sqrt{(1-x^2)}}{x^2\sqrt{(1-x^4)}}$.

30. $y = \sqrt{\dfrac{1-x^2}{(1+x^2)^3}}$. $\qquad \dfrac{dy}{dx} = \dfrac{2x^3-4x}{(1-x^2)^{\frac{1}{2}}(1+x^2)^{\frac{5}{2}}}$.

31. $y = x(a^2+x^2)\sqrt{(a^2-x^2)}$. $\qquad \dfrac{dy}{dx} = \dfrac{a^4 + a^2x^2 - 4x^4}{\sqrt{(a^2-x^2)}}$.

32. $y = \dfrac{x^{2n}}{(1+x^2)^n}$. $\qquad \dfrac{dy}{dx} = \dfrac{2n\,x^{2n-1}}{(1+x^2)^{n+1}}$.

5 33. $y = \dfrac{1-x}{\sqrt{(1+x^2)}}$. $\qquad \dfrac{dy}{dx} = -\dfrac{1+x}{(1+x^2)^{\frac{3}{2}}}$.

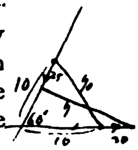
34. $y = \dfrac{x^3}{x+\sqrt{(1+x^2)}}$. *See Example* 23. $\qquad \dfrac{dy}{dx} = \dfrac{4x^4+3x^2}{\sqrt{(x^2+1)}} - 4x^3$.

35. $y = \dfrac{\sqrt{(1+x^2)} + \sqrt{(1-x^2)}}{\sqrt{(1+x^2)} - \sqrt{(1-x^2)}}$. $= \frac{1}{x^2}(1+\sqrt{1-x^4})$ $\qquad \dfrac{dy}{dx} = -\dfrac{2}{x^3}\left[1 + \dfrac{1}{\sqrt{(1-x^4)}}\right]$.

36. $y = \dfrac{x}{\sqrt{(x^2+a^2)} - a}$. $\qquad \dfrac{dy}{dx} = -\dfrac{a}{x^2}\left[1 + \dfrac{a}{\sqrt{(x^2+a^2)}}\right]$.

37. $y = \dfrac{x\sqrt{(a+x)}}{\sqrt{a} - \sqrt{(a-x)}}$. $= \sqrt{a}\sqrt{a+x}+\sqrt{a^2-x^2}$ $\qquad \dfrac{dy}{dx} = \dfrac{\sqrt{a}}{2\sqrt{(a+x)}} - \dfrac{x}{\sqrt{(a^2-x^2)}}$.

38. Two locomotives are moving along two straight lines of railway which intersect at an angle of 60° ; one is approaching the intersection at the rate of 25 miles an hour, and the other is receding from it at the rate of 30 miles an hour; find the rate per hour at which they are separating from each other when each is 10 miles from the intersection.



$y^2 = (10 - v_1 t)^2 + (10 + v_2 t)^2 - \cancel{2}(10 - v_1 t)(10 + v_2 t)$ ~~cos 60~~

$\frac{dy}{dt} = \frac{-(10 - v_1 t)(v_2 + 2v_1) + (10 + v_2 t)(v_1 + 2v_2)}{2y}$, make $t = 0$

$2\frac{1}{2}$ miles.

39. A street-crossing is 10 ft. from a street-lamp situated directly above the curbstone, which is 60 ft. from the vertical walls of the opposite buildings. If a man is walking across to the opposite side of the street at the rate of 4 miles an hour, at what rate per hour does his shadow move upon the walls—(α) when he is 5 ft. from the curbstone? (β) when he is 20 ft. from the curbstone?

(α) 96 miles; (β) 6 miles.

40. Assuming the volume of a tree to be proportional to the cube of its diameter, and that the latter increases uniformly; find the ratio of the rate of its volume when the diameter is 6 inches to the rate when the diameter is 3 ft.

$V = kD^3$ $\quad \frac{D_1^2}{D_2^2} = \frac{(\frac{1}{2})^2}{3^2} = \frac{1}{36}$.

$V' = 3kD^2$

41. If an ingot of silver in the form of a parallelopiped expand $\frac{1}{1000}$ part of each of its linear dimensions for each degree of temperature, at what rate per degree of temperature is its volume increasing when the sides are respectively 2, 3, and 6 inches?

If x denote a side, dx may be assumed to denote the rate per degree of temperature. $\frac{27}{250}$ of a cubic inch.

42. Prove generally that, if the coefficient of expansion of each linear dimension of a solid is k, its coefficient of expansion in volume is $3k$.

Solution :—

Let x denote any side; then, if V denote the volume, we shall have $V = cx^3$; c being a constant dependent on the shape of the body.

Therefore $\qquad dV = 3cx^2\,dx$;

or, since $\qquad dx = kx$,

$$dV = 3kcx^3 = 3kV.$$

43. Wine is poured into a conical glass 3 inches in height at a uniform rate, filling the glass in 8 seconds. At what rate is the surface

rising at the end of 1 second? At what rate when the surface reaches the brim?

Solution:—

Let h denote the height of the glass (3 inches), v_1 its entire volume, v and x the volume and height corresponding to the time t, and a the time required to fill the glass (8 seconds); then

$$\frac{x^3}{h^3}=\frac{v}{v_1}, \qquad \text{and} \qquad \frac{v}{v_1}=\frac{t}{a},$$

therefore
$$x=h\frac{v^{\frac{1}{3}}}{v_1^{\frac{1}{3}}}=\frac{h}{a^{\frac{1}{3}}}t^{\frac{1}{3}};$$

whence
$$\frac{dx}{dt}=\frac{h}{3a^{\frac{1}{3}}}t^{-\frac{2}{3}},$$

$$\left.\frac{dx}{dt}\right]_1=\frac{h}{3a^{\frac{1}{3}}}=\tfrac{1}{2}, \qquad \text{and} \qquad \left.\frac{dx}{dt}\right]_8=\frac{h}{12a^{\frac{1}{3}}}=\tfrac{1}{8}.$$

44. A person walking along a bridge at the rate of 3 miles per hour is 20 yards above, and vertically over, another in a boat, which is gliding from under the arch at the rate of 8 miles per hour in a course at right angles to the bridge; at what rate per hour are they separating at the end of 3 minutes? 8·541 miles.

CHAPTER III.

THE DIFFERENTIATION OF TRANSCENDENTAL FUNCTIONS.

VI.

The Logarithmic Function.

39. IN this chapter, the formulas for the differentiation of the simple transcendental functions are to be established.

We begin by deducing the differential of the logarithmic function, employing the method exemplified in Art. 29.

The symbol $\log x$ is used in this article to denote the logarithm of x to any base, and $\log_b x$ is used when we wish to designate a particular base b.

Let $\quad z = mx, \qquad \therefore \qquad \log z = \log m + \log x,$

differentiating by Art. 21,

$$dz = m\,dx, \qquad \text{and} \qquad d(\log z) = d(\log x);$$

whence
$$\frac{d(\log z)}{dz} = \frac{d(\log x)}{m\,dx}$$

Multiplying by $z = mx$, to eliminate m, we obtain

$$z\frac{d(\log z)}{dz} = x\frac{d(\log x)}{dx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The derivatives, $\dfrac{d(\log z)}{dz}$ and $\dfrac{d\,(\log x)}{dx}$, are, by Art. 23, func-

tions of z and of x respectively, independent of the values of dz and dx; moreover, equation (1) is true for all values of x and z, these quantities being entirely independent of each other, since the arbitrary ratio m has been eliminated. Hence, in equation (1), one of the quantities, x or z, may be assumed to have a fixed value, while the other is variable; whence it follows that the members of this equation have a fixed value independent of the values of x and z; we therefore write

$$x\,\frac{d(\log x)}{dx} = \text{a constant.} \quad \ldots\ldots \quad (2)$$

This constant, although independent of x, may be dependent on the value of the base of the system of logarithms under consideration. Denoting the base of the system by b, we therefore denote the constant by B, and write equation (2) thus,—

$$d(\log_b x) = \frac{B\,dx}{x}. \quad \ldots\ldots \quad (3)$$

40. *To determine the value of B*, we establish a relation between two values of the base and the corresponding values of this unknown quantity.

Denoting another value of the base by a, and the corresponding value of the unknown constant by A, we have

$$d(\log_a x) = \frac{A\,dx}{x}. \quad \ldots\ldots \quad (4)$$

The relation sought may now be obtained by differentiating, by means of (3) and (4), the identical equation

$$\log_a x = \log_a b \log_b x,^{*}. \quad \ldots\ldots \quad (5)$$

* This identity is most readily obtained thus,—by definition

$$x = b^{\log_b x};$$

thus obtaining $$\frac{A\,dx}{x} = \log_a b \frac{B\,dx}{x},$$

or $$B \log_a b = A,$$

hence $$\log_a b^B = A,$$

that is, A is the logarithm to the base a of b^B; whence we have

$$b^B = a^A. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Now, it is obvious that the value of a^A cannot depend upon b, hence equation (6) shows that the value of b^B likewise cannot depend upon b; b^B must, therefore, have a value entirely independent of b. Denoting this constant value by ϵ, we write

$$b^B = \epsilon. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Adopting this constant as a base, and taking the logarithms of each member of equation (7), we have

$$B \log_\epsilon b = 1,$$

whence $$B = \frac{1}{\log_\epsilon b}.$$

Introducing this value of B in equation (3), we obtain

$$d(\log_b x) = \frac{dx}{\log_\epsilon b \,.\, x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (g)$$

In this equation, the differential of a logarithm to any given base is expressed by the aid of the unknown constant ϵ.

41. The constant ϵ is employed as the base of a system of

---

taking the logarithm to the base a of each member, we have

$$\log_a x = \log_b x \, \log_a b.$$

logarithms, sometimes called ***natural*** or ***hyperbolic***, but more commonly ***Napierian*** logarithms, from the name of the inventor of logarithms. Hence ϵ is known as the *Napierian base*.

Putting $b = \epsilon$ in formula (g) we derive

$$d(\log_\epsilon x) = \frac{dx}{x}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (g')$$

The logarithms employed in analytical investigations are almost exclusively Napierian. Whenever it is necessary, for the purpose of obtaining numerical results, these logarithms may be expressed in terms of the common tabular logarithms by means of the formula,

$$\log_{10} x = \log_{10}\epsilon \, \log_\epsilon x,$$

which is derived from equation (5), Art. 40, by writing 10 for a and ϵ for b. The value of the constant $\log_{10}\epsilon$ will be computed in a subsequent chapter.

Hereafter, whenever the symbol log is employed without the subscript, $\log_\epsilon$ is to be understood.

The Logarithmic Curve.

42. The curve, corresponding to the equation

$$y = \log_\epsilon x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is called the *logarithmic curve*.

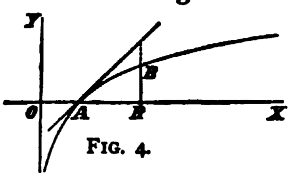


FIG. 4.

The shape of this curve is indicated in Fig. 4. It passes through the point A whose coordinates are (1, 0), since

$$\log 1 = 0.$$

Since we have, from formula (g'),

$$\tan\phi = \frac{dy}{dx} = \frac{1}{x}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

the value of $\tan\phi$ at the point A is unity, and therefore the tangent line at this point cuts the axis of x at an angle of $45°$, as in the diagram. We have from equation (2),

when	$x > 1$	$\tan\phi < 1$,
and when	$x < 1$	$\tan\phi > 1$;

the curve, therefore, lies below this tangent, as shown in Fig. 4.

The point $(\epsilon, 1)$ is a point of the curve; let B, Fig. 4, be this point, then OR will represent the Napierian base, and $BR = 1$. Since

$$OA = 1, \qquad \text{and} \qquad AR > BR,$$

$$OR > 2;$$

that is, the Napierian base ϵ is somewhat greater than 2.

The quantity ϵ is incommensurable: the method of computing its value to any required degree of accuracy is given in a subsequent chapter.

Logarithmic Differentiation.

43. The differential of the Napierian logarithm of the variable x, that is the expression $\frac{dx}{x}$, is called the *logarithmic differential* of x.

When x has a negative value, the expression $\log x$ has no real value; in this case, however, $\log(-x)$ is real, and we have

$$d[\log(-x)] = \frac{d(-x)}{-x} = \frac{dx}{x}.$$

This expression therefore, in the case of a negative quantity, is identical with the logarithmic differential of the positive quantity having the same numerical value.

44. The process of taking logarithms and differentiating the result is called *logarithmic differentiation.* By means of this method, all the formulas for the differentiation of algebraic functions may be derived.

In the following logarithmic equations, it is to be understood that that sign is taken in each case which will render the logarithm real.

By differentiating the formulas,—

$$\log(\pm xy) = \log(\pm x) + \log(\pm y),$$

$$\log\left(\pm\frac{x}{y}\right) = \log(\pm x) - \log(\pm y),$$

$$\log(\pm x^n) = n\log(\pm x),$$

we obtain

$$\frac{d(xy)}{xy} = \frac{dx}{x} + \frac{dy}{y},$$

$$\frac{y}{x}\,d\left(\frac{x}{y}\right) = \frac{dx}{x} - \frac{dy}{y},$$

$$\frac{d(x^n)}{x^n} = n\,\frac{dx}{x}.$$

These formulas are evidently equivalent to (*c*), (*e*), and (*f*), of which we thus have an independent proof.

45. The method of logarithmic differentiation may frequently be used with advantage in finding the derivatives of complicated algebraic expressions. For example, let us take

$$u = \frac{\sqrt{(2x)}\,(1-x^2)^{\frac{3}{4}}}{(x-2)^{\frac{2}{3}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Hence, we derive

$$\log u = \tfrac{1}{2}\log(2x) + \tfrac{3}{4}\log(1 - x^2) - \tfrac{2}{3}\log(x - 2), \quad . \quad . \quad (2)$$

differentiating,

$$\frac{du}{u\,dx} = \frac{1}{2x} - \frac{3}{2}\,\frac{x}{1 - x^2} - \frac{2}{3}\,\frac{1}{x - 2}, \quad . \quad . \quad . \quad . \quad (3)$$

adding and reducing,

$$\frac{du}{u\,dx} = \frac{-8x^3 + 24x^2 - x - 6}{6(1 - x^2)(x - 2)x};$$

therefore

$$\frac{du}{dx} = \frac{-8x^3 + 24x^2 - x - 6}{3(2x)^{\frac{1}{2}}(1 - x^2)^{\frac{1}{4}}(x - 2)^{\frac{5}{3}}}.$$

For certain values of x, one or more of the quantities whose logarithms appear in equation (2) become negative. When this is the case these logarithms should, strictly speaking, be replaced by the logarithms of the numerical values of the quantities in question; this change however would not affect the form of equation (3). See Art. 43.

Exponential Functions.

46. An *exponential function* is an expression in which an exponent is a function of the independent variable. The quantity affected by the exponent may be constant or variable. In the first case, let the function be denoted by

$$y = a^x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If a is negative, a^x cannot denote a continuously varying quantity. We therefore exclude the case in which a has a negative value, and regard a^x as a continuously varying positive quantity.

Taking Napierian logarithms of both members of equation (1), we have

$$\log y = x \log a;$$

differentiating by (g'),

$$\frac{dy}{y} = \log a \,.\, dx\,;$$

hence $$dy = \log a \,.\, y\,dx,$$

or $$d(a^x) = \log a \,.\, a^x\,dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad (h)$$

Exponential functions of the form ϵ^x are of frequent occurrence. Putting $a = \epsilon$ in formula (h), we have

$$d(\epsilon^x) = \epsilon^x\,dx\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (h')$$

hence the derivative of the function ϵ^x is identical with the function itself. This function is the inverse of the Napierian logarithm; it has been proposed to denote it by the symbol $\exp x$.

47. When both the exponent and the quantity affected by it are variable, the method of logarithmic differentiation may be employed. Thus, if the given function be

$$z = (nx)^{x^2},$$

we shall have $$\log z = x^2 \log(nx)\,;$$

differentiating, $$\frac{dz}{z} = x^2\frac{dx}{x} + 2x\log(nx)\,dx,$$

hence $$d[(nx)^{x^2}] = (nx)^{x^2}x[1 + 2\log(nx)]\,dx.$$

Examples VI.

1. Given the function $y = \log_b x$; show that $\left.\frac{dy}{dx}\right]_\epsilon = \frac{\log_b \epsilon}{\epsilon}$, and hence prove that the tangent to the corresponding curve, at the point whose abscissa is ϵ, passes through the origin.

Put $a = x = \epsilon$ in equation 5, Art. 40.

2. $y = x^n \log x.$ $\qquad \dfrac{dy}{dx} = x^{n-1}(1 + n\log x).$

3. $y = \log(\log x).$ $\qquad \dfrac{dy}{dx} = \dfrac{1}{x\log x}.$

4. $y = \log[\log(a + bx^n)].$ $\qquad \dfrac{dy}{dx} = \dfrac{nbx^{n-1}}{(a + bx^n)\log(a + bx^n)}.$

5. $y = \sqrt{x} - \log(\sqrt{x} + 1).$ $\qquad \dfrac{dy}{dx} = \dfrac{1}{2(\sqrt{x} + 1)}.$

6 6. $y = \log\dfrac{\sqrt{a} + \sqrt{x}}{\sqrt{a} - \sqrt{x}}.$ $\qquad \dfrac{dy}{dx} = \dfrac{\sqrt{a}}{(a - x)\sqrt{x}}.$

Put in the form, $\log(\sqrt{a} + \sqrt{x}) - \log(\sqrt{a} - \sqrt{x}).$

7. $y = \log[\sqrt{(x-a)} + \sqrt{(x-b)}].$ $\qquad \dfrac{dy}{dx} = \dfrac{1}{2\sqrt{[(x-a)(x-b)]}}.$

8. $y = \log[x + \sqrt{(x^2 \pm a^2)}].$ $\qquad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{(x^2 \pm a^2)}}.$

7 × 9. $y = \log\dfrac{x}{\sqrt{(1 + x^2)}}.$ $\qquad \dfrac{dy}{dx} = \dfrac{1}{x(1 + x^2)}.$

10. $y = \log\dfrac{\sqrt{(1+x)} + \sqrt{(1-x)}}{\sqrt{(1+x)} - \sqrt{(1-x)}}.$ $\qquad \dfrac{dy}{dx} = -\dfrac{1}{x\sqrt{(1-x^2)}}.$

11. $y = \log[x + \sqrt{(a^2 - x^2)}].$ $\qquad \dfrac{dy}{dx} = \dfrac{\sqrt{(a^2 - x^2)} - x}{\sqrt{(a^2 - x^2)}[x + \sqrt{(a^2 - x^2)}]}.$

12. $y = \log\dfrac{x}{\sqrt{(x^2 + a^2)} - x}.$ $\qquad \dfrac{dy}{dx} = \dfrac{1}{x} + \dfrac{1}{\sqrt{(x^2 + a^2)}}.$

13. $y = \log[\sqrt{(1 + x^2)} + \sqrt{(1 - x^2)}].$ $\qquad \dfrac{dy}{dx} = \dfrac{1}{x} - \dfrac{1}{x\sqrt{(1 - x^4)}}.$

14. $y = \log(x - a) - \dfrac{a(2x - a)}{(x - a)^2}.$ $\qquad \dfrac{dy}{dx} = \dfrac{x^2 + a^2}{(x - a)^3}.$

15. $y = a^{x^2}$. $\qquad \dfrac{dy}{dx} = 2 \log a . a^{x^2} x.$

16. $y = \varepsilon^{\frac{1}{1+x}}$. $\qquad \dfrac{dy}{dx} = - \dfrac{1}{(1+x)^2} \cdot \varepsilon^{\frac{1}{1+x}}.$

17. $y = \varepsilon^x(1 - x^3)$. $\qquad \dfrac{dy}{dx} = \varepsilon^x(1 - 3x^2 - x^3).$

18. $y = (x - 3)\,\varepsilon^{2x} + 4x\,\varepsilon^x$. $\qquad \dfrac{dy}{dx} = (2x - 5)\,\varepsilon^{2x} + 4(x + 1)\varepsilon^x.$

19. $y = \dfrac{\varepsilon^x - \varepsilon^{-x}}{\varepsilon^x + \varepsilon^{-x}}$. $\qquad \dfrac{dy}{dx} = \dfrac{4}{(\varepsilon^x + \varepsilon^{-x})^2}.$

20. $y = b^{a^x}$. $\qquad \dfrac{dy}{dx} = \log a \,.\, \log b \,.\, b^{a^x} .\, a^x.$

21. $y = a^{x^n}$. $\qquad \dfrac{dy}{dx} = n\,a^{x^n} .\, x^{n-1} .\, \log a.$

22. $y = \dfrac{x}{\varepsilon^x - 1}$. $\qquad \dfrac{dy}{dx} = \dfrac{\varepsilon^x(1 - x) - 1}{(\varepsilon^x - 1)^2}.$

23. $y = \log(\varepsilon^x + \varepsilon^{-x})$. $\qquad \dfrac{dy}{dx} = \dfrac{\varepsilon^x - \varepsilon^{-x}}{\varepsilon^x + \varepsilon^{-x}}.$

24. $y = a^{\log x}$. $\qquad \dfrac{dy}{dx} = \dfrac{1}{x} \log_\varepsilon a \,.\, a^{\log x}.$

9 25. $y = \log \dfrac{\varepsilon^x}{1 + \varepsilon^x}$. $\qquad \dfrac{dy}{dx} = \dfrac{1}{1 + \varepsilon^x}.$ ✓

26. $y = x^x$. $\qquad \dfrac{dy}{dx} = x^x(1 + \log x).$

27. $y = x^{\log x}$. $\qquad \dfrac{dy}{dx} = \dfrac{2 \log x}{x}\, x^{\log x}.$

28. $y = \varepsilon^{x^x}$. $\qquad \dfrac{dy}{dx} = \varepsilon^{x^x} .\, x^x(1 + \log x).$

29. $y = x^{\frac{1}{x}}$. $\qquad \dfrac{dy}{dx} = x^{\frac{1}{x}} \cdot \dfrac{1 - \log x}{x^2}$.

30. $y = \varepsilon^{\varepsilon^x}$. $\qquad \dfrac{dy}{dx} = \varepsilon^{\varepsilon^x} \cdot \varepsilon^x$.

31. $y = x^{\varepsilon^x}$. $\qquad \dfrac{dy}{dx} = x^{\varepsilon^x} \cdot \varepsilon^x \dfrac{1 + x \log x}{x}$.

32. $y = a^x (x \log a - 1)$. $\qquad \dfrac{dy}{dx} = (\log a)^2 \, x a^x$.

33. $y = 2\varepsilon^{\sqrt{x}} (x^{\frac{3}{2}} - 3x + 6x^{\frac{1}{2}} - 6)$. $\qquad \dfrac{dy}{dx} = x \varepsilon^{\sqrt{x}}$.

34. $y = \dfrac{(x-1)^{\frac{5}{2}}}{(x-2)^{\frac{3}{4}} (x-3)^{\frac{7}{3}}}$. $\qquad \dfrac{dy}{dx} = -\dfrac{(x-1)^{\frac{3}{2}} (7x^2 + 30x - 97)}{12 (x-2)^{\frac{7}{4}} (x-3)^{\frac{10}{3}}}$.

See Art. 45.

35. $y = \dfrac{\sqrt{[ax(x-3a)]}}{\sqrt{(x-4a)}}$. $\qquad \dfrac{dy}{dx} = \dfrac{\sqrt{a}\,(x^2 - 8ax + 12a^2)}{2[x(x-3a)]^{\frac{1}{2}} (x-4a)^{\frac{3}{2}}}$.

36. $y = \dfrac{(x+1)^{\frac{1}{2}} (x+3)^{\frac{9}{2}}}{(x+2)^4}$. $\qquad \dfrac{dy}{dx} = \dfrac{x^2 (x+3)^{\frac{7}{2}}}{(x+2)^5 (x+1)^{\frac{1}{2}}}$.

37. $y = \dfrac{(x-2)^9}{(x-1)^{\frac{5}{2}} (x-3)^{\frac{11}{2}}}$. $\qquad \dfrac{dy}{dx} = \dfrac{(x-2)^8 (x^2 - 7x + 1)}{(x-1)^{\frac{7}{2}} (x-3)^{\frac{13}{2}}}$.

38. $y = \dfrac{(x^2 - 2x + 2)^{\frac{1}{4}} (x^2+1)^{\frac{3}{2}}}{(x+1)^{\frac{7}{2}}}$.

$$\frac{dy}{dx} = \frac{(8x^3 - 21x^2 + 26x - 15)(x^2+1)^{\frac{1}{2}}}{2(x^2 - 2x + 2)^{\frac{3}{4}} (x+1)^{\frac{9}{2}}}.$$

39. $y = \dfrac{\varepsilon^{x\sqrt{(x^2-1)}}}{x + \sqrt{(x^2-1)}}$. $\qquad \dfrac{dy}{dx} = \dfrac{2\sqrt{(x^2-1)}\,\varepsilon^{x\sqrt{(x^2-1)}}}{x + \sqrt{(x^2-1)}}$.

40. $y = \left(\dfrac{x}{1 + \sqrt{(1-x^2)}}\right)^n$. $\qquad \dfrac{dy}{dx} = \dfrac{ny}{x\sqrt{(1-x^2)}}$.

41. $u = \frac{x}{\sqrt{(1-x^2)}}\left(\frac{x}{1+\sqrt{(1-x^2)}}\right)^n.$

$$\frac{du}{dx} = \left(\frac{x}{1+\sqrt{(1-x^2)}}\right)^n \frac{1+n\sqrt{(1-x^2)}}{(1-x^2)^{\frac{3}{2}}}.$$

Put $\left(\frac{x}{1+\sqrt{(1-x^2)}}\right)^n = y$, *and use the result obtained in Ex.* 40.

VII.

The Trigonometric or Circular Functions.

48. In deriving the differentials of the trigonometric functions of a variable angle, we employ the *circular measure* of the angle, and denote it by θ. Thus, let s denote the length of the arc subtending the angle in the circle whose radius is a, then

$$\theta = \frac{s}{a}.$$

In Fig. 5, let OA be a fixed line, and OP an equal line rotating about the origin O; then P will describe the circle whose equation (the coordinates being rectangular) is

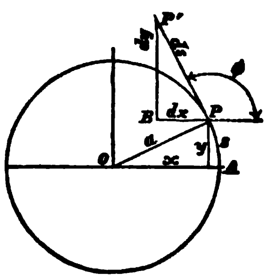


FIG. 5.

$$x^2 + y^2 = a^2.$$

The velocity of the point P is the rate of s, and (see Art. 17) is denoted by $\frac{ds}{dt}$, which has a positive value when P moves so as to increase θ. Let PP', taken in the direction of the motion of P, represent ds; then, according to the definition given in Art. 25, PP' is a tangent line, and PB and BP' will represent dx and dy, as in Art. 26.

49. We have first to show that the line PP', which is a tangent to the curve according to the general definition (Art. 25), is perpendicular to the radius.

Differentiating the equation of the circle, we have

$$x\,dx + y\,dy = 0;$$

whence
$$\tan\phi = \frac{dy}{dx} = -\frac{x}{y}.$$

Now (see Fig. 5),
$$\frac{y}{x} = \tan\theta,$$

therefore,
$$\tan\phi = -\cot\theta = \tan(\theta \pm \tfrac{1}{2}\pi),$$

or,
$$\phi = \theta \pm \tfrac{1}{2}\pi;$$

hence the tangent line is perpendicular to the radius. Assuming ϕ to be the angle between the positive directions of x and ds, we have

$$\phi = \theta + \tfrac{1}{2}\pi.$$

The Sine and the Cosine.

50. From Fig. 5, it is evident that

$$\sin\theta = \frac{y}{a}, \qquad \text{and} \qquad \cos\theta = \frac{x}{a};$$

therefore
$$d(\sin\theta) = \frac{dy}{a}, \quad \text{and} \quad d(\cos\theta) = \frac{dx}{a}. \quad . \quad . \quad . \quad (1)$$

In equations (1) we have to express dy and dx in terms of θ and $d\theta$.

Again, from the figure, we have

$$dy = \sin\phi . ds, \qquad \text{and} \qquad dx = \cos\phi . ds\,;*$$

substituting in equations (1), we obtain

$$d(\sin\theta) = \sin\phi\,\frac{ds}{a}, \quad \text{and} \quad d(\cos\theta) = \cos\phi\,\frac{ds}{a}. \quad . \quad . \quad . \quad (2)$$

Since $\phi = \theta + \frac{1}{2}\pi$, and $\frac{s}{a} = \theta$,

$$\sin\phi = \cos\theta, \quad \cos\phi = -\sin\theta, \quad \text{and} \quad \frac{ds}{a} = d\theta.$$

Substituting these values in equations (2), we obtain

$$d(\sin\theta) = \cos\theta\, d\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (i)$$

and

$$d(\cos\theta) = -\sin\theta\, d\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (j)$$

The Tangent and the Cotangent.

51. The differential of $\tan\theta$ is found by applying formula (e) to the equation

$$\tan\theta = \frac{\sin\theta}{\cos\theta};$$

thus,

$$d(\tan\theta) = \frac{\cos\theta\, d(\sin\theta) - \sin\theta\, d(\cos\theta)}{\cos^2\theta},$$

or

$$d(\tan\theta) = \frac{d\theta}{\cos^2\theta} = \sec^2\theta\, d\theta. \quad . \quad . \quad . \quad . \quad (k)$$

*In Fig. 5, dx is negative; but, ϕ being in the second quadrant, $\cos\phi$ is likewise negative.

The differential of $\cot\theta$ is found by applying formula (k) to the equation

$$\cot\theta = \tan(\tfrac{1}{2}\pi - \theta);$$

whence $$d(\cot\theta) = -\frac{d\theta}{\sin^2\theta} = -\operatorname{cosec}^2\theta\, d\theta. \quad . \quad . \quad . \quad (l)$$

The Secant and the Cosecant.

52. The differential of $\sec\theta$ is found by applying formula (d) to the equation

$$\sec\theta = \frac{1}{\cos\theta};$$

whence $$d(\sec\theta) = \frac{\sin\theta\, d\theta}{\cos^2\theta} = \sec\theta\tan\theta\, d\theta. \quad . \quad . \quad (m)$$

The differential of $\operatorname{cosec}\theta$ is found by applying formula (m) to the equation

$$\operatorname{cosec}\theta = \sec(\tfrac{1}{2}\pi - \theta);$$

whence $$d(\operatorname{cosec}\theta) = -\frac{\cos\theta\, d\theta}{\sin^2\theta} = -\operatorname{cosec}\theta\cot\theta\, d\theta. \quad . \quad (n)$$

The Versed-Sine.

53. The *versed-sine* is defined by the equation

$$\operatorname{vers}\theta = 1 - \cos\theta;$$

therefore $$d(\operatorname{vers}\theta) = \sin\theta\, d\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (o)$$

Examples VII.

1. The value of $d(\sin\theta)$ being given, derive that of $d(\cos\theta)$ from the formula

$$\cos\theta = \sin(\tfrac{1}{2}\pi - \theta);$$

also from the identity

$$\cos^2\theta = 1 - \sin^2\theta.$$

2. From the identity $\sec^2\theta = 1 + \tan^2\theta$, derive the differential of $\sec\theta$.

3. From the identity $\sin 2\theta = 2\sin\theta\cos\theta$, derive another by taking derivatives. $\cos 2\theta = \cos^2\theta - \sin^2\theta.$

4. From the identity $\sin(\theta \pm \tfrac{1}{4}\pi) = \tfrac{1}{2}\sqrt{2}\,(\sin\theta \pm \cos\theta)$, derive another by taking derivatives. $\cos(\theta \pm \tfrac{1}{4}\pi) = \tfrac{1}{2}\sqrt{2}\,(\cos\theta \mp \sin\theta).$

5. Prove the formulas :—

$$d(\log\sin\theta) = -d(\log\operatorname{cosec}\theta) = \cot\theta\,d\theta;$$

$$d(\log\cos\theta) = -d(\log\sec\theta) = -\tan\theta\,d\theta;$$

$$d(\log\tan\theta) = -d(\log\cot\theta) = (\tan\theta + \cot\theta)\,d\theta.$$

6. Obtain an identity by taking derivatives of both members of the equation

$$\tan\tfrac{1}{2}\theta = \frac{1-\cos\theta}{\sin\theta}.$$

$$\tfrac{1}{2}\sec^2\tfrac{1}{2}\theta = \frac{1-\cos\theta}{\sin^2\theta}.$$

7. $y = \theta + \sin\theta\cos\theta.$ $\qquad \dfrac{dy}{d\theta} = 2\cos^2\theta$

8. $y = \sin\theta - \tfrac{1}{3}\sin^3\theta.$ $\qquad \dfrac{dy}{d\theta} = \cos^3\theta.$

9. $y = \dfrac{\sin\theta}{\sqrt{(\cos\theta)}}.$ $\qquad \dfrac{dy}{d\theta} = \dfrac{1+\cos^2\theta}{2(\cos\theta)^{\frac{3}{2}}}.$

10. $y = \frac{1}{3}\tan^3\theta - \tan\theta + \theta.$ $\qquad \dfrac{dy}{d\theta} = \tan^4\theta.$

11. $y = \frac{1}{3}\tan^3\theta + \tan\theta.$ $\qquad \dfrac{dy}{d\theta} = \sec^4\theta.$

12. $y = \sin e^x.$ $\qquad \dfrac{dy}{dx} = e^x \cos e^x.$

13. $y = x\sin x^2.$ $\qquad \dfrac{dy}{dx} = \sin x^2 + 2x^2 \cos x^2.$

14. $y = a^{\sin x}.$ $\qquad \dfrac{dy}{dx} = \log a \,.\, a^{\sin x}\cos x.$

15. $y = \tan^2\theta + \log(\cos^2\theta).$ $\qquad \dfrac{dy}{d\theta} = 2\tan^3\theta.$

16. $y = \log(\tan\theta + \sec\theta).$ $\qquad \dfrac{dy}{d\theta} = \sec\theta.$

17. $y = \log\tan(\frac{1}{4}\pi + \frac{1}{2}\theta).$ $\qquad \dfrac{dy}{d\theta} = \dfrac{1}{\cos\theta}.$

18. $y = x + \log\cos(\frac{1}{4}\pi - x).$ $\qquad \dfrac{dy}{dx} = \dfrac{2}{1 + \tan x}.$

19. $y = \log\sqrt{(\sin x)} + \log\sqrt{(\cos x)}.$ $\qquad \dfrac{dy}{dx} = \cot 2x.$

20. $y = \sin n\theta\,(\sin\theta)^n.$ $\qquad \dfrac{dy}{d\theta} = n(\sin\theta)^{n-1}\sin(n+1)\theta.$

21. $y = \dfrac{\sin x}{1 + \tan x}.$ $\qquad \dfrac{dy}{dx} = \dfrac{\cos^3 x - \sin^3 x}{(\sin x + \cos x)^2}.$

22. $y = e^{ax}\cos bx.$ $\qquad \dfrac{dy}{dx} = e^{ax}(a\cos bx - b\sin bx).$

23. $y = \log\sqrt{\dfrac{a\cos x - b\sin x}{a\cos x + b\sin x}}.$ $\qquad \dfrac{dy}{dx} = \dfrac{-ab}{a^2\cos^2 x - b^2\sin^2 x}.$

24. $y = \tan \epsilon^{\frac{1}{x}}$ $\qquad \dfrac{dy}{dx} = -\dfrac{\epsilon^{\frac{1}{x}} \sec^2 \epsilon^{\frac{1}{x}}}{x^2}.$

25. $y = \epsilon^{ax}(a \sin x - \cos x).$ $\qquad \dfrac{dy}{dx} = (a^2 + 1)\epsilon^{ax} \sin x.$

26. $y = \epsilon^{x}(\cos x - \sin x).$ $\qquad \dfrac{dy}{dx} = -2\epsilon^{x} \sin x.$

27. $y = \epsilon^{-a^2x^2} \cos rx.$ $\qquad \dfrac{dy}{dx} = -\epsilon^{-a^2x^2}(2a^2 x \cos rx + r \sin rx).$

28. $y = \dfrac{(\sin nx)^m}{(\cos mx)^n}.$ $\qquad \dfrac{dy}{dx} = \dfrac{mn(\sin nx)^{m-1} \cos(mx - nx)}{(\cos mx)^{n+1}}.$

29. $y = \tan \sqrt{(1 - x)}.$ $\qquad \dfrac{dy}{dx} = \dfrac{-[\sec \sqrt{(1-x)}]^2}{2\sqrt{(1-x)}}.$

30. $y = x^{\sin x}.$ $\qquad \dfrac{dy}{dx} = x^{\sin x}\left(\cos x . \log x + \dfrac{\sin x}{x}\right).$

31. $y = \sin(\log nx).$ $\qquad \dfrac{dy}{dx} = \dfrac{\cos(\log nx)}{x}.$

32. $y = \sin(\sin x).$ $\qquad \dfrac{dy}{dx} = \cos x . \cos(\sin x).$

33. $y = \dfrac{2}{\sin^2 x \cos x} - \dfrac{3 \cos x}{\sin^2 x} + 3 \log \tan \dfrac{x}{2}.$ $\qquad \dfrac{dy}{dx} = \dfrac{2}{\sin^3 x \cos^2 x}.$

34. Given $x = r \cos \theta$, and $y = r \sin \theta$, prove that

$$dy \sin \theta + dx \cos \theta = dr,$$

and

$$dy \cos \theta - dx \sin \theta = r\, d\theta.$$

35. From $x = r \cos \theta$, and $y = r \sin \theta$, deduce

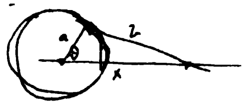
$$(dx)^2 + (dy)^2 = (dr)^2 + r^2(d\theta)^2.$$

36. The crank of a small steam-engine is 1 foot in length, and

revolves uniformly at the rate of two turns per second, the connecting rod being 5 ft. in length; find the velocity per second of the piston when the crank makes an angle of 45° with the line of motion of the piston-rod; also when the angle is 135°, and when it is 90°.

Solution:—

Let a, b, and x denote respectively the crank, the connecting-rod, and the variable side of the triangle; and let θ denote the angle between a and x.



We easily deduce

$$x = a\cos\theta + \sqrt{(b^2 - a^2\sin^2\theta)};$$

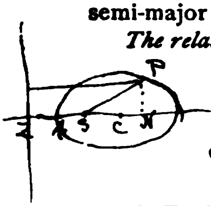
whence $$\frac{dx}{dt} = -\left(a\sin\theta + \frac{a^2\sin\theta\cos\theta}{\sqrt{(b^2 - a^2\sin^2\theta)}}\right)\frac{d\theta}{dt}.$$

In this case, $\frac{d\theta}{dt} = 4\pi$, $a = 1$, and $b = 5$.

When $\theta = 45°$, $\frac{dx}{dt} = -\frac{16\pi\sqrt{2}}{7}$ ft. ✓

✓37. An elliptical cam revolves at the rate of two turns per second about a horizontal axis passing through one of the foci, and gives a reciprocating motion to a bar moving in vertical guides in a line with the centre of rotation: denoting by θ the angle between the vertical and the major axis, find the velocity per second with which the bar is moving when $\theta = 60°$, the eccentricity of the ellipse being $\frac{1}{2}$, and the semi-major axis 9 inches. Also find the velocity when $\theta = 90°$.

The relation between θ and the radius vector is expressed by the equation



$SP = e.SZ = e(r\cos\theta + CZ - CS) = e(r\cos\theta + \frac{a}{e} - ae)$

$$\therefore \quad r = \frac{a(1-e^2)}{1 - e\cos\theta}.$$

$\frac{dr}{dt} = -\frac{ae(1-e^2)\sin\theta}{(1-e\cos\theta)^2}\frac{d\theta}{dt}$ When $\theta = 60°$, $\frac{dr}{dt} = -12\sqrt{3}\pi$ inches. ✓

38. Find an expression in terms of its azimuth for the rate at which the altitude of a star is increasing.

Solution:—

Let h denote the altitude and A the azimuth of the star, p its polar distance, t the hour angle, and L the latitude of the observer; the formulas of spherical trigonometry give

$$\sin h = \sin L\cos p + \cos L\sin p\cos t, \quad \ldots\ldots \quad (1)$$

and $$\sin p\sin t = \sin A\cos h. \quad \ldots\ldots\ldots \quad (2)$$

Differentiating (1), p and L being constant,

$$\cos h \frac{dh}{dt} = -\cos L \sin p \sin t,$$

whence, substituting the value of $\sin p \sin t$, from equation (2),

$$\frac{dh}{dt} = -\cos L \sin A.$$

It follows that $\frac{dh}{dt}$ is greatest when $\sin A$ is numerically greatest; that is, when the star is on the prime vertical. In the case of a star that never reaches the prime vertical, the rate is greatest when A is greatest.

39. Trace the curves $y = \sin x$, $y = \tan x$, and $y = \sec x$, determining in each case the value of $\tan \phi$, for the point at which the curve cuts the axis of y.

VIII.

The Inverse Circular Functions.

54. It is shown in Trigonometry that, if

$$x = \sin \theta,$$

the expressions

or $n\pi + (-1)^n\theta$, where n is any $\pm$ integer $\quad 2n\pi + \theta \qquad$ and $\qquad (2n+1)\pi - \theta, \quad . \quad . \quad (1)$

in which n denotes zero or any integer, include all the arcs of which the sine is x; hence each of these arcs is a value of the *inverse function*

$$\sin^{-1} x.$$

Among these values, there is always one, and *only one*, which falls between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$; since, while the arc

passes from the former of these values to the latter, the sine passes from -1 to $+1$; that is, it passes once through all its possible values.

Let θ, in the expressions (1), denote this value, which we shall call the *primary* value of the function.

55. In a similar manner, if

$$x = \cos\theta,$$

each of the arcs included in the expression

$$2n\pi \pm \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

is a value of the inverse function

$$\cos^{-1} x.$$

One of these values, and only one, falls between 0 and π; since, while the arc passes from the former of these values to the latter, its cosine passes from $+1$ to -1; that is, once through all its possible values. In expression (2), let θ denote this value, which we shall call the *primary* value of this function.

56. In the case of the function

$$\operatorname{cosec}^{-1} x,$$

the definition of the *primary* value that was adopted in the case of $\sin^{-1} x$, and the same general expressions (1) for the values of the function, are applicable.

In the case of the function

$$\sec^{-1} x,$$

the definition of the *primary* value adopted in the case of

$\cos^{-1} x$ and expression (2) for the general value of the function are applicable.

Finally, in the case of each of the functions

$$\tan^{-1} x \qquad \text{and} \qquad \cot^{-1} x$$

the *primary* value (θ) is taken between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$, and the general expression for the value of the function is

$$n\pi + \theta. \quad \dots\dots \quad (3)$$

The Inverse Sine and the Inverse Cosine.

57. To find the differential of the inverse sine, let

$$\theta = \sin^{-1} x;$$

then $\qquad x = \sin\theta, \qquad$ and $\qquad dx = \cos\theta\, d\theta,$

or
$$d\theta = \frac{dx}{\cos\theta}.$$

Now, $\qquad \cos\theta = \pm\sqrt{(1 - \sin^2\theta)} = \pm\sqrt{(1 - x^2)},$

hence
$$d(\sin^{-1} x) = \frac{dx}{\pm\sqrt{(1 - x^2)}}. \quad \dots\dots \quad (1)$$

If θ denotes the primary value of this function; that is, the value between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$, $\cos\theta$ is positive. Hence the upper sign in this ambiguous result belongs to the differential of the primary value of the function; it is therefore usual to write

$$d(\sin^{-1} x) = \frac{dx}{\sqrt{(1 - x^2)}}. \quad \dots\dots \quad (p)$$

Since we have, from expressions (1), Art. 54,

$$d(2n\pi + \theta) = d\theta, \qquad \text{and} \qquad d[(2n+1)\pi - \theta] = -d\theta,$$

it is evident that the positive sign in equation (1) belongs not only to the differential of the primary value of $\sin^{-1}x$, but likewise to the differentials of all the values included in $2n\pi+\theta$; and that the negative sign belongs to the differentials of the values of $\sin^{-1}x$ included in $(2n+1)\pi-\theta$.

58. Similarly, if

$$\theta=\cos^{-1}x, \qquad x=\cos\theta;$$

whence

$$d\theta=\frac{dx}{-\sin\theta},$$

or

$$d(\cos^{-1}x)=\frac{dx}{\mp\sqrt{(1-x^2)}} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If θ denote the primary value of the function which in this case is between 0 and π, $\sin\theta$ is positive; hence the upper sign in this ambiguous result belongs to the differential of the primary value. It is therefore usual to write

$$d(\cos^{-1}x)=\frac{dx}{-\sqrt{(1-x^2)}}. \quad . \quad . \quad . \quad . \quad . \quad (q)$$

Since, from expression (2), Art. 55, we have

$$d(2n\pi\pm\theta)=\pm d\theta;$$

it is evident that the upper and lower signs in equation (1) correspond to the upper and lower signs, respectively, in the general expression $2n\pi\pm\theta$.

The Inverse Tangent and the Inverse Cotangent.

59. Let

$$\theta=\tan^{-1}x, \qquad \text{then} \qquad x=\tan\theta;$$

differentiating, we derive,

$$d\theta=\frac{dx}{\sec^2\theta}.$$

But $\sec^2\theta = 1 + \tan^2\theta = 1 + x^2$, therefore,

$$d(\tan^{-1} x) = \frac{dx}{1+x^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (r)$$

No ambiguity arises in the value of the differential of this function; since, from expression (3), Art. 56, we have

$$d(n\pi + \theta) = d\theta.$$

Similarly, putting

$$\theta = \cot^{-1} x,$$

we derive
$$d(\cot^{-1} x) = -\frac{dx}{1+x^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (s)$$

The Inverse Secant and Inverse Cosecant.

60. Let

$$\theta = \sec^{-1} x, \qquad \text{then} \qquad x = \sec\theta;$$

differentiating, we derive

$$d\theta = \frac{dx}{\sec\theta \tan\theta}.$$

But $\sec\theta = x$, and $\tan\theta = \pm\sqrt{(\sec^2\theta - 1)} = \pm\sqrt{(x^2 - 1)}$, therefore,

$$d(\sec^{-1} x) = \frac{dx}{\pm x\sqrt{(x^2 - 1)}}.$$

If x is positive, and if θ denotes the primary value of the function, $\tan\theta$ is positive. Hence it is usual to write

$$d(\sec^{-1} x) = \frac{dx}{x\sqrt{(x^2 - 1)}}. \quad . \quad . \quad . \quad . \quad . \quad (t)$$

When x is negative, if θ denotes the primary value of the function, which in this case is in the second quadrant, $\tan\theta$ is negative; consequently the radical must be taken with the negative sign. Hence, since x is also negative, the value of the differential is positive, when the arc is taken in the second quadrant.

In like manner we derive

$$d(\operatorname{cosec}^{-1}x) = -\frac{dx}{x\sqrt{(x^2-1)}}. \quad . \quad . \quad . \quad . \quad (u)$$

Similar remarks apply also to this differential when x is negative.

The Inverse Versed-Sine.

61. Let

$$\theta = \operatorname{vers}^{-1}x, \qquad \text{then} \qquad x = \operatorname{vers}\theta = 1 - \cos\theta,$$

and $\quad 1 - x = \cos\theta, \qquad \therefore \qquad d\theta = \dfrac{dx}{\sin\theta}.$

But $\sin\theta = \sqrt{(1-\cos^2\theta)} = \sqrt{(2x - x^2)}$, therefore,

$$d(\operatorname{vers}^{-1}x) = \frac{dx}{\sqrt{(2x-x^2)}}. \quad . \quad . \quad . \quad . \quad . \quad (v)$$

Illustrative Examples.

62. It is sometimes advantageous to transform a given function before differentiating, by means of one of the following formulas:—

$$\sin^{-1}\frac{\alpha}{\beta} = \operatorname{cosec}^{-1}\frac{\beta}{\alpha}, \quad \cos^{-1}\frac{\alpha}{\beta} = \sec^{-1}\frac{\beta}{\alpha}, \quad \tan^{-1}\frac{\alpha}{\beta} = \cot^{-1}\frac{\beta}{\alpha}.$$

Thus, let
$$y = \tan^{-1}\frac{\varepsilon^x \cos x}{1 + \varepsilon^x \sin x},$$

then
$$y = \cot^{-1}(\varepsilon^{-x}\sec x + \tan x).$$

By formula (s),

$$\frac{dy}{dx} = -\frac{\varepsilon^{-x}\sec x \tan x - \varepsilon^{-x}\sec x + \sec^2 x}{\sec^2 x + 2\varepsilon^{-x}\sec x \tan x + \varepsilon^{-2x}\sec^2 x},$$

multiplying both terms by $\varepsilon^{2x}\cos^2 x$,

$$\frac{dy}{dx} = \frac{\varepsilon^x(\cos x - \sin x - \varepsilon^x)}{1 + 2\varepsilon^x \sin x + \varepsilon^{2x}}.$$

63. Trigonometric substitutions may sometimes be employed with advantage. Thus, let

$$y = \tan^{-1}\frac{x}{\sqrt{(1+x^2)} + 1}.$$

If in this example we put $x = \tan\theta$, we have

$$y = \tan^{-1}\frac{\tan\theta}{\sec\theta + 1} = \tan^{-1}\frac{\sin\theta}{1 + \cos\theta}$$

$$= \tan^{-1}(\tan\tfrac{1}{2}\theta) = \tfrac{1}{2}\theta = \tfrac{1}{2}\tan^{-1}x.$$

Examples VIII.

1. Derive from (p), (r), and (t) the formulas:—

$$d\left(\sin^{-1}\frac{x}{a}\right) = \frac{dx}{\sqrt{(a^2 - x^2)}};$$

$$d\left(\tan^{-1}\frac{x}{a}\right) = \frac{a\,dx}{a^2 + x^2};$$

$$d\left(\sec^{-1}\frac{x}{a}\right)=\frac{a\,dx}{x\sqrt{(x^2-a^2)}}.$$

2. Derive $d(\sec^{-1}x)$ from the equation $\sec^{-1}x=\cos^{-1}\frac{1}{x}$.

3. Derive $d\left(\cot^{-1}\frac{x}{a}\right)$ from the equation $\cot^{-1}\frac{x}{a}=\tan^{-1}\frac{a}{x}$.

4. $y=\sin^{-1}(2x^2)$. $\frac{dy}{dx}=\frac{4x}{\sqrt{(1-4x^4)}}$. ✓

5. $y=\sin^{-1}(\cos x)$. $\frac{dy}{dx}=-1$.

6. $y=\sin(\cos^{-1}x)$. $\frac{dy}{dx}=-\frac{x}{\sqrt{(1-x^2)}}$.

7. $y=\sin^{-1}(\tan x)$. $\frac{dy}{dx}=\frac{\sec^2 x}{\sqrt{(1-\tan^2 x)}}$.

8. $y=\cos^{-1}(2\cos x)$. $\frac{dy}{dx}=-\frac{2\sin x}{\sqrt{(1-4\cos^2 x)}}$.

9. $y=x\sin^{-1}x+\sqrt{(1-x^2)}$. $\frac{dy}{dx}=\sin^{-1}x$.

10. $y=\tan^{-1}\epsilon^x$. $\frac{dy}{dx}=\frac{1}{\epsilon^x+\epsilon^{-x}}$.

11. $y=(x^2+1)\tan^{-1}x-x$. $\frac{dy}{dx}=2x\tan^{-1}x$. ✓

12. $y=a^2\sin^{-1}\frac{x}{a}+x\sqrt{(a^2-x^2)}$. $\frac{dy}{dx}=2\sqrt{(a^2-x^2)}$.

13. $y=\tan^{-1}\frac{mx}{1-x^2}$. $\frac{dy}{dx}=\frac{m(1+x^2)}{1+(m^2-2)x^2+x^4}$.

14. $y=\sin^{-1}\frac{x+1}{\sqrt{2}}$. $\frac{dy}{dx}=\frac{1}{\sqrt{(1-2x-x^2)}}$.

15. $y = \tan^{-1}\frac{x}{\sqrt{(1-x^2)}}$. $\qquad \frac{dy}{dx} = \frac{1}{\sqrt{(1-x^2)}}$.

16. $y = \sec^{-1}\frac{a}{\sqrt{(a^2-x^2)}}$. $\qquad \frac{dy}{dx} = \frac{1}{\sqrt{(a^2-x^2)}}$.

17. $y = \sin^{-1}\frac{x}{\sqrt{(x^2+a^2)}}$. $\qquad \frac{dy}{dx} = \frac{a}{a^2+x^2}$.

18. $y = \sin^{-1}\sqrt{(\sin x)}$. $\qquad \frac{dy}{dx} = \frac{1}{2}\sqrt{(1+\operatorname{cosec} x)}$.

19. $y = \sqrt{(1-x^2)}\sin^{-1}x - x$. $\qquad \frac{dy}{dx} = -\frac{x\sin^{-1}x}{\sqrt{(1-x^2)}}$.

20. $y = \tan^{-1}\frac{m+x}{1-mx}$. $\qquad \frac{dy}{dx} = \frac{1}{1+x^2}$.

21. $y = \cos^{-1}\frac{1-x^2}{1+x^2}$. $\qquad \frac{dy}{dx} = \frac{2}{1+x^2}$.

22. $y = \tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$. $\qquad \frac{dy}{dx} = \frac{1}{2}$.

23. $y = \frac{x\sin^{-1}x}{\sqrt{(1-x^2)}} + \log\sqrt{(1-x^2)}$. $\qquad \frac{dy}{dx} = \frac{\sin^{-1}x}{(1-x^2)^{\frac{3}{2}}}$.

24. $y = (x+a)\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{(ax)}$. $\qquad \frac{dy}{dx} = \tan^{-1}\sqrt{\frac{x}{a}}$.

25. $y = \tan^{-1}\frac{x\sqrt{3}}{2+x}$. $\qquad \frac{dy}{dx} = \frac{\sqrt{3}}{2(x^2+x+1)}$.

26. $y = \tan^{-1}\frac{2cx+b}{\sqrt{(4ac-b^2)}}$. $\qquad \frac{dy}{dx} = \frac{1}{2}\frac{\sqrt{(4ac-b^2)}}{a+bx+cx^2}$.

27. $y = \epsilon^{\sin^{-1}x}$. $\qquad \frac{dy}{dx} = \frac{\epsilon^{\sin^{-1}x}}{\sqrt{(1-x^2)}}$.

28. $y = x^{\sec^{-1}x}$. $\qquad \frac{dy}{dx} = x^{\sec^{-1}x}\left(\frac{\log x}{x\sqrt{(x^2-1)}} + \frac{\sec^{-1}x}{x}\right)$.

29. $y = x\,\epsilon^{\tan^{-1}x}$. $\qquad \dfrac{dy}{dx} = \epsilon^{\tan^{-1}x}\left(\dfrac{x^2+x+1}{x^2+1}\right).$

30. $y = \epsilon^{(1+x^2)\tan^{-1}x}$. $\qquad \dfrac{dy}{dx} = (1 + 2x\tan^{-1}x)\,\epsilon^{(1+x^2)\tan^{-1}x}.$

31. $y = \cos^{-1}\dfrac{x^2-2}{x^2}$. $\qquad \dfrac{dy}{dx} = -\dfrac{3}{x\sqrt{(x^2-1)}}.$

32. $y = \tan^{-1}[x + \sqrt{(1-x^2)}]$. $\qquad \dfrac{dy}{dx} = \dfrac{\sqrt{(1-x^2)}-x}{2\sqrt{(1-x^2)}\,[1+x\sqrt{(1-x^2)}]}.$

33. $y = \sin^{-1}\dfrac{b + a\cos x}{a + b\cos x}$. $\qquad \dfrac{dy}{dx} = -\dfrac{\sqrt{(a^2-b^2)}}{a + b\cos x}.$

34. $y = \sec^{-1}\dfrac{x\sqrt{5}}{2\sqrt{(x^2+x-1)}}$. $\qquad \dfrac{dy}{dx} = \dfrac{1}{x\sqrt{(x^2+x-1)}}.$

35. $y = \tan^{-1}\dfrac{3a^2x - x^3}{a^3 - 3ax^2}$. $\qquad \dfrac{dy}{dx} = \dfrac{3a}{a^2+x^2}.$

36. $y = \tan^{-1}\dfrac{\sqrt{(b^2-a^2)}\sin x}{a + b\cos x}$. $\qquad \dfrac{dy}{dx} = \dfrac{\sqrt{(b^2-a^2)}}{b + a\cos x}.$

37. $y = \sin^{-1}\dfrac{x\sqrt{(a-b)}}{\sqrt{[a(1+x^2)]}}$. $\qquad \dfrac{dy}{dx} = \dfrac{\sqrt{(a-b)}}{(1+x^2)\sqrt{(a+bx^2)}}.$

38. Trace the curves $y = \sin^{-1}x$, $y = \tan^{-1}x$, and $y = \sec^{-1}x$, indicating the portions which correspond to the primary values of the functions.

IX.

Differentials of Functions of Two Variables.

64. The formulas already deduced enable us to differentiate any function of two variables, expressed by elementary functional symbols; the application of these formulas is, how-

ever, sometimes facilitated by a general principle which will now be shown to be applicable to such functions.

The formulas mentioned above involve differential factors of the first degree only. It follows, therefore, that the differentials resulting from their application consist of terms each of which contains the first power of the differential of one of the variables. In other words, if

$$u = f(x,y),$$

$$du = \phi(x,y)\,dx + \psi(x,y)\,dy. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now, if y were constant, we should have $dy = 0$, and the value of du would reduce to that of the first term in the right-hand member of (1); hence this term may be found by *differentiating* u *on the supposition that* y *is constant*, and in like manner the second term can be found by differentiating u on the supposition that x is constant. The sum of the results thus obtained is therefore the required value of du.

65. As an example, let

$$z = u^v.$$

Were v constant, we should have for the value of dz, by formula (f), Art. 37,

$$v u^{v-1}\,du;$$

and, were u constant, we should have, by formula (h), Art. 46,

$$\log u \,.\, u^v\,dv;$$

whence, adding these results,

$$dz = u^{v-1}(v\,du + u \log u\,dv).$$

Although this result has been obtained on the supposition that u and v are independent variables, it is evident that any two functions of a single variable may be substituted for u and v. Thus, if

$$u = nx \qquad \text{and} \qquad v = x^2,$$

we have
$$z = (nx)^{x^2},$$

and, on substituting,

$$dz = (nx)^{x^2 - 1}(x^2 n\,dx + nx \log(nx) \,.\, 2x\,dx),$$
$$= x(nx)^{x^2}[1 + 2\log(nx)]dx,$$

which is identical with the expression obtained in Art. 47, for the differential of this function.

The Derivatives of Implicit Functions.

66. When x and y are connected by an equation such that y cannot be made an explicit function of x, the value of the derivative $\dfrac{dy}{dx}$ cannot be expressed in terms of x; it can, however, be expressed in terms of x and y, and the numerical value of the derivative, for any known simultaneous values of x and y, can be determined. Thus, if the given equation is

$$xy^3 - 3x^2y + 6y^2 + 2x = 0, \quad \ldots \ldots \quad (1)$$

we obtain

$$y^3\,dx + 3xy^2\,dy - 6xy\,dx - 3x^2\,dy + 12y\,dy + 2dx = 0,$$

whence
$$\frac{dy}{dx} = -\frac{y^3 - 6xy + 2}{3(xy^2 - x^2 + 4y)}. \quad \ldots \ldots \quad (2)$$

Now, observing that equation (1) is satisfied by the values, $x = 2$ and $y = 1$, we find, by substitution in (2),

$$\left.\frac{dy}{dx}\right]_{2,1} = \tfrac{8}{5},$$

in which the subscripts denote the given values of x and y.

67. The expression for the derivative may sometimes be simplified by means of the given equation. Thus, if we have

$$x^{y^n} = y^{x^n}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or

$$y^n \log x = x^n \log y, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

on differentiating, we obtain

$$\frac{y^n}{x} dx + n y^{n-1} \log x \, dy = \frac{x^n}{y} dy + n x^{n-1} \log y \, dx.$$

Whence
$$\frac{dy}{dx} = \frac{y(y^n - n x^n \log y)}{x(x^n - n y^n \log x)},$$

and, by substituting from equation (2),

$$\frac{dy}{dx} = \frac{y^{n+1}}{x^{n+1}} \cdot \frac{1 - n \log x}{1 - n \log y}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In this example, equation (1) is evidently satisfied by $x = y = a$ for all values of a, and equation (3) gives, except when $a = \epsilon^{\frac{1}{n}}$,

$$\left.\frac{dy}{dx}\right]_{a,a} = 1.$$

When $a = \epsilon^{\frac{1}{n}}$, this derivative takes the form $\frac{0}{0}$, and its value can be determined by the method given in Art. 117.

Examples IX.

1. $u = xy\, e^{x+2y}$. $\qquad du = e^{x+2y}[y(1+x)\,dx + x(1+2y)\,dy]$.

2. $u = \log\tan\dfrac{x}{y}$. $\qquad du = 2\dfrac{y\,dx - x\,dy}{y^2 \sin 2\dfrac{x}{y}}$.

3. $u = \log\tan^{-1}\dfrac{x}{y}$. $\qquad du = \dfrac{y\,dx - x\,dy}{(x^2+y^2)\tan^{-1}\dfrac{x}{y}}$.

4. $u = \dfrac{\sqrt{x} + \sqrt{y}}{x+y}$.

$$du = \frac{[y - x - 2\sqrt{(xy)}]\sqrt{y}\,dx + [x - y - 2\sqrt{(xy)}]\sqrt{x}\,dy}{2\sqrt{(xy)}\,(x+y)^2}.$$

5. $u = \dfrac{e^x y}{(x^2+y^2)^{\frac{1}{2}}}$. $\qquad du = \dfrac{y e^x\,dx}{(x^2+y^2)^{\frac{1}{2}}} + \dfrac{x(x\,dy - y\,dx)\,e^x}{(x^2+y^2)^{\frac{3}{2}}}$.

6. $u = \tan^{-1}\dfrac{x-y}{x+y}$. $\qquad du = \dfrac{y\,dx - x\,dy}{x^2+y^2}$.

7. $u = \sqrt{\dfrac{x^2-y^2}{x^2+y^2}}$. $\qquad du = \dfrac{2xy(y\,dx - x\,dy)}{(x^2+y^2)^{\frac{3}{2}}\sqrt{(x^2-y^2)}}$.

8. $u = \log\dfrac{x + \sqrt{(x^2-y^2)}}{x - \sqrt{(x^2-y^2)}}$. $\qquad du = \dfrac{2(y\,dx - x\,dy)}{y\sqrt{(x^2-y^2)}}$.

9. Given $x = r\cos\theta$, and $y = r\sin\theta$; eliminate θ and find dr; also eliminate r and find $d\theta$.

$$dr = \frac{x\,dx + y\,dy}{\sqrt{(x^2+y^2)}}, \text{ and } d\theta = \frac{x\,dy - y\,dx}{x^2+y^2}.$$

10. Given $x^2(y-1) + y^2(x+1) = 1$; find an expression in terms of x and y for $\dfrac{dy}{dx}$, and also its numerical values when $y = 2$.

Deduce the corresponding values of x from the original equation.

$$\left.\frac{dy}{dx}\right]_{-1,\,2} = -6, \text{ and } \left.\frac{dy}{dx}\right]_{-7;\,2} = -\frac{6}{23}.$$

11. Given $xy^2 + x^2y - 2 = 0$; determine the value of $\dfrac{dy}{dx}$ by the method exemplified in Art. 66, also by making y an explicit function of x; and find the values of the derivative corresponding to $x = 1$.

$$\frac{dy}{dx} = -\frac{y^2 + 2xy}{2xy + x^2} = -\tfrac{1}{2} \pm \frac{x^2 - 4}{2x\sqrt{(x^4 + 8x)}}.$$

$$\left.\frac{dy}{dx}\right]_{1,\,1} = -1, \text{ and } \left.\frac{dy}{dx}\right]_{1,\,-2} = 0.$$

12. $\tan^{-1}\dfrac{x-a}{x+a} - \tan^{-1}\dfrac{y-a}{y+a} = b.$ $\qquad \dfrac{dy}{dx} = \dfrac{y^2 + a^2}{x^2 + a^2}.$

13. $y = 1 + xe^y.$ $\qquad \dfrac{dy}{dx} = \dfrac{e^y}{2 - y}.$

14. $(x - y)y^n = x + y.$ $\qquad \dfrac{dy}{dx} = \dfrac{-2y^2}{n(x^2 - y^2) - 2xy}.$

15. $(x^2 + y^2)^2 = a^2x^2 - b^2y^2.$ $\qquad \dfrac{dy}{dx} = \dfrac{[a^2 - 2(x^2 + y^2)]x}{[b^2 + 2(x^2 + y^2)]y}.$

16. $ye^{ny} = ax^m.$ $\qquad \dfrac{dy}{dx} = \dfrac{my}{x(1 + ny)}.$

17. $y^3 - 3y\sin^{-1}x + x^3 = 0.$ $\qquad \dfrac{dy}{dx} = 3y\dfrac{y - x^2(1 - x^2)^{\frac{1}{2}}}{(2y^3 - x^3)(1 - x^2)^{\frac{1}{2}}}.$

18. $y \sin nx - ae^{nx+y} = 0.$ $\qquad \dfrac{dy}{dx} = \dfrac{ny}{1 - y}(1 - \cot nx).$

19. $y\tan^{-1}x - y^2 + x^2 = 0.$ $\qquad \dfrac{dy}{dx} = \dfrac{y(y + 2x + 2x^3)}{(1 + x^2)(y^2 + x^2)}.$

20. $y = \tan(x + y).$ $\qquad \dfrac{dy}{dx} = -\dfrac{1 + y^2}{y^2}.$

Miscellaneous Examples.

1. $y = \dfrac{x}{\sqrt{(1 + x)}}.$ $\qquad \dfrac{dy}{dx} = \dfrac{x + 2}{2(1 + x)^{\frac{3}{2}}}.$

2. $y = \sqrt{\dfrac{a^2 - x^2}{b^2 - x^2}}$. $\qquad \dfrac{dy}{dx} = \dfrac{(a^2 - b^2)\,x}{(a^2 - x^2)^{\frac{1}{2}}(b^2 - x^2)^{\frac{3}{2}}}$.

3. $y = \dfrac{\sqrt{(a + x)}}{\sqrt{a} + \sqrt{x}}$. $\qquad \dfrac{dy}{dx} = \dfrac{\sqrt{a}\,(\sqrt{x} - \sqrt{a})}{2\sqrt{x}\,\sqrt{(a + x)}\,(\sqrt{a} + \sqrt{x})^2}$.

4. $y = (\sqrt{x} - 2\sqrt{a})\,\sqrt{(\sqrt{a} + \sqrt{x})}$. $\qquad \dfrac{dy}{dx} = \dfrac{3}{4\sqrt{(\sqrt{a} + \sqrt{x})}}$.

5. $y = \dfrac{(x - 1)(\epsilon^x + 1)\,\epsilon^x}{\epsilon^x - 1}$. $\qquad \dfrac{dy}{dx} = \dfrac{\epsilon^x(x\,\epsilon^{2x} - 2x\,\epsilon^x + 2\epsilon^x - x)}{(\epsilon^x - 1)^2}$.

6. $y = x^{y+x}$. $\qquad \dfrac{dy}{dx} = \dfrac{x^{y+x}(x^{y+x-1} + \log x + 1)}{1 - x^{y+x}\log x}$.

7. $y = x^{x^n}$ $\qquad \dfrac{dy}{dx} = (n\log x + 1)\,x^{x^n}\,x^{n-1}$.

8. $y = x^{x^x}$ $\qquad \dfrac{dy}{dx} = x^{x^x}\,x^x\left[(\log x)^2 + \log x + \dfrac{1}{x}\right]$.

9. $y = \log\dfrac{(1 + x^2)^{\frac{1}{4}}}{(1 + x)^{\frac{1}{2}}} + \frac{1}{2}\tan^{-1}x$. $\qquad \dfrac{dy}{dx} = \dfrac{x}{(1 + x)(1 + x^2)}$.

10. $y = \log\left(\dfrac{1 + x}{1 - x}\right)^{\frac{1}{4}} - \frac{1}{2}\tan^{-1}x$. $\qquad \dfrac{dy}{dx} = \dfrac{x^2}{1 - x^4}$.

11. $y = \log[x + \sqrt{(x^2 - a^2)}] + \sec^{-1}\dfrac{x}{a}$. $\qquad \dfrac{dy}{dx} = \dfrac{1}{x}\sqrt{\left(\dfrac{x + a}{x - a}\right)}$.

12. $y = \dfrac{2\sin^{-1}x}{\sqrt{(1 - x^2)}} + \log\dfrac{1 - x}{1 + x}$. $\qquad \dfrac{dy}{dx} = \dfrac{2x\sin^{-1}x}{(1 - x^2)^{\frac{3}{2}}}$.

13. $y = \dfrac{(1 + 3x + 3x^2)^{\frac{1}{3}}}{x}$. $\qquad \dfrac{dy}{dx} = -\dfrac{(1 + x)^2}{x^2(1 + 3x + 3x^2)^{\frac{2}{3}}}$.

14. $y = a\log\dfrac{a + \sqrt{(a^2 - x^2)}}{x} - \sqrt{(a^2 - x^2)}$.

$$\frac{dy}{dx} = -\frac{\sqrt{(a^2 - x^2)}}{x}.$$

15. $y = \dfrac{(1 - x^2)^{\frac{3}{2}} \sin^{-1} x}{x}$.

$$\frac{dy}{dx} = \frac{1 - x^2}{x} - \frac{1 + 2x^2}{x^2} \sqrt{(1 - x^2)} \sin^{-1} x.$$

16. $y = \log \sqrt{\dfrac{1 - \cos x}{1 + \cos x}}$. $\qquad \dfrac{dy}{dx} = \dfrac{1}{\sin x}$.

17. $y = \tan^{-1} \left[\sqrt{\dfrac{a - b}{a + b}} \cdot \tan \dfrac{x}{2} \right]$. $\qquad \dfrac{dy}{dx} = \dfrac{\sqrt{(a^2 - b^2)}}{2(a + b \cos x)}$.

18. $y = \sec^{-1} \dfrac{1}{2x^2 - 1}$. $\qquad \dfrac{dy}{dx} = - \dfrac{2}{\sqrt{(1 - x^2)}}$.

19. $y = \cos^{-1} \dfrac{x^{2n} - 1}{x^{2n} + 1}$. $\qquad \dfrac{dy}{dx} = - \dfrac{2n x^{n-1}}{x^{2n} + 1}$.

20. $y = a \cos^{-1} \dfrac{a - x}{b} - \sqrt{[b^2 - (a - x)^2]}$.

$$\frac{dy}{dx} = \frac{x}{\sqrt{[b^2 - (a - x)^2]}}.$$

21. $y = \cos^{-1} x - 2 \sqrt{\dfrac{1 - x}{1 + x}}$. $\qquad \dfrac{dy}{dx} = \dfrac{\sqrt{(1 - x)}}{(1 + x)^{\frac{3}{2}}}$.

22. $y = \dfrac{ax - 1}{\sqrt{(1 + x^2)}} \epsilon^{a \tan^{-1} x}$. $\qquad \dfrac{dy}{dx} = \dfrac{(1 + a^2) x}{(1 + x^2)^{\frac{3}{2}}} \epsilon^{a \tan^{-1} x}$.

Use logarithmic differentials.

23. $y = \dfrac{(5x)^2 - 480x + 288}{125 (4 - 5x)^2} + \dfrac{12}{125} \log (4 - 5x)$. $\qquad \dfrac{dy}{dx} = \dfrac{5x^3}{(5x - 4)^3}$.

24. $y = \sin^{-1} \dfrac{x \tan \alpha}{\sqrt{(a^2 - x^2)}}$. $\qquad \dfrac{dy}{dx} = \dfrac{a^2 \tan \alpha}{a^2 - x^2} \cdot \dfrac{1}{\sqrt{(a^2 - x^2 \sec^2 \alpha)}}$.

25. $y = \cos^{-1} \sqrt{\left(\dfrac{a^2 - x^2}{b^2 - x^2} \right)}$. $\qquad \dfrac{dy}{dx} = \dfrac{x \sqrt{(b^2 - a^2)}}{(b^2 - x^2) \sqrt{(a^2 - x^2)}}$.

26. $y = \tan^{-1}\dfrac{\sqrt{(a+bx)}}{\sqrt{(b-a)}}$. $\qquad \dfrac{dy}{dx} = \dfrac{\sqrt{(b-a)}}{2(1+x)\sqrt{(a+bx)}}$.

27. $y = \log\tan\dfrac{x}{2} - \dfrac{\cos x}{\sin^2 x}$. $\qquad \dfrac{dy}{dx} = \dfrac{2}{\sin^3 x}$.

28. $y = \log\dfrac{1 + x\sqrt{2} + x^2}{1 - x\sqrt{2} + x^2} + 2\tan^{-1}\dfrac{x\sqrt{2}}{1-x^2}$. $\qquad \dfrac{dy}{dx} = \dfrac{4\sqrt{2}}{1+x^4}$.

29. $y = \dfrac{1}{(1+x)^4}\left(\dfrac{1}{x} + \dfrac{125}{12} + \dfrac{65x}{3} + \dfrac{35x^2}{2} + 5x^3\right) + 5\log\dfrac{x}{1+x}$.

$$\frac{dy}{dx} = -\frac{1}{x^2(1+x)^5}.$$

30. $y = \log\dfrac{1+x}{1-x} + \frac{1}{2}\log\dfrac{1+x+x^2}{1-x+x^2} + \sqrt{3}\tan^{-1}\dfrac{x\sqrt{3}}{1-x^2}$.

$$\frac{dy}{dx} = \frac{6}{1-x^6}.$$

31. $y = (1+x^2)^{\frac{m}{2}}\sin(m\tan^{-1}x)$.

$$\frac{dy}{dx} = m(1+x^2)^{\frac{m-1}{2}}\cos[(m-1)\tan^{-1}x].$$

32. $y = \log\dfrac{(x-1)^2}{x^2+x+1} - 2\sqrt{3}\tan^{-1}\dfrac{2x+1}{\sqrt{3}}$. $\qquad \dfrac{dy}{dx} = \dfrac{6x}{x^3-1}$.

33. Given $u = xz + a\sin z + az\cos z$, and $x = a - a\cos z$; prove that

$$\frac{du}{dx} = \left(\frac{2a-x}{x}\right)^{\frac{1}{2}}.$$

CHAPTER IV.

SUCCESSIVE DIFFERENTIATION.

X.

Velocity and Acceleration.

68. IF the variable quantity x represent the distance of a point, moving in a straight line, from a fixed origin taken on the line, the rate of x will represent the velocity of the point.

Denoting this velocity by v_x we have, in accordance with the definition given in Art. 17,

$$v_x = \frac{dx}{dt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In this expression the arbitrary interval of time dt is regarded as constant, while dx, and consequently v_x, are in general variable. Differentiating equation (1) we have, since dt is constant,

$$dv_x = \frac{d(dx)}{dt}.$$

The differential of dx, denoted above by $d(dx)$, is called the *second differential* of x; it is usually written in the abbreviated form d^2x, and read "d-second x." The rate of v_x is therefore expressed thus:—

$$\frac{dv_x}{dt} = \frac{d^2x}{(dt)^2}.$$

The rate of the velocity of a point is called its *acceleration*, and is usually denoted by α; hence we write

$$\alpha_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

the marks of parenthesis being usually omitted in the denominator of this expression.

69. When the space x described by a moving point is a given function of the time t, the derivative of this function is, by equation (1), an expression for the velocity in terms of t. The derivative of the latter expression, which is called the *second derivative* of x, is therefore, by equation (2), an expression for the acceleration in terms of t.

A *positive* value of the acceleration α indicates an *algebraic increase* of the velocity v, whether the latter be positive or negative; and, on the other hand, a negative value of α indicates an *algebraic decrease* of the velocity.

70. As an illustration, let x denote the space which a body falling freely describes in the time t. A well-known mechanical formula gives

$$x = \tfrac{1}{2}gt^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Hence we derive

$$v_x = \frac{dx}{dt} = gt, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and

$$\alpha_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} = g. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In this case, therefore, the acceleration is constant and positive, and accordingly v_x, which is likewise positive, is numerically increasing.

71. When the velocity is given in terms of x, the acceleration can readily be expressed in terms of the same variable, as in the following example.

Given $\quad v_x = 2 \sin x;$

whence $\quad \dfrac{dv_x}{dt} = 2 \cos x \dfrac{dx}{dt};$

that is, $\quad \alpha_x = 2 \cos x . v_x = 4 \cos x \sin x = 2 \sin 2x.$

The general expression for α_x, when v_x is given in terms of x, is

$$\alpha_x = \frac{dv_x}{dt} = \frac{dv_x}{dx}\frac{dx}{dt} = v_x \frac{dv_x}{dx} = \frac{1}{2}\frac{d(v_x^2)}{dx} \quad . \quad . \quad . \quad . \quad (1)$$

Component Velocities and Accelerations.

72. When the motion of a point is not rectilinear but is nevertheless confined to a plane, its position is referred to coordinate axes; the coordinates, x and y, are evidently functions of t, and the derivatives $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$, which denote the rates of these variables, are called the *component* or *resolved velocities* in the directions of the axes. Denoting these component velocities by v_x and v_y, we have

$$v_x = \frac{dx}{dt}, \text{ and } v_y = \frac{dy}{dt}.$$

Again, denoting by s the actual space described, as measured from some fixed point of the path, s will likewise be a function of t, and the derivative $\dfrac{ds}{dt}$ will denote the actual velocity of the point. (Compare Art. 48.) Now, the axes being rectangular, and ϕ denoting the inclination of the direction of the motion to the axis of x, we have

$$dx = ds \cos \phi, \text{ and } dy = ds \sin \phi.$$

Hence, $\quad \dfrac{dx}{dt} = \dfrac{ds}{dt} \cos \phi, \text{ and } \dfrac{dy}{dt} = \dfrac{ds}{dt} \sin \phi;$

or $$v_x = v\cos\phi, \text{ and } v_y = v\sin\phi.$$

Squaring and adding,

$$v_x^2 + v_y^2 = v^2.$$

The last equation enables us to determine from the component velocities the actual velocity in the curve.

73. If we represent the accelerations of the resolved motions in the directions of the axes by α_x and α_y, we shall have, by Art. 68,

$$\alpha_x = \frac{d^2x}{dt^2} \text{ and } \alpha_y = \frac{d^2y}{dt^2}.$$

These accelerations, α_x and α_y, will be positive when the resolved motions are *accelerated in the positive directions of the corresponding axes;* that is, when they increase a positive resolved velocity, or numerically decrease a negative resolved velocity.

Examples X.

1. The space in feet described in the time t by a point moving in a straight line is expressed by the formula

$$x = 48t - 16t^2;$$

find the acceleration, and the velocity at the end of $2\frac{1}{2}$ seconds; also find the value of t for which $v = 0$.

$\alpha = -32$; $v = 0$, when $t = 1\frac{1}{2}$.

2. If the space described in t seconds be expressed by the formula

$$x = 10\log\frac{4}{4+t};$$ $v = -\frac{10}{4+t}$, $\alpha = \frac{10}{(4+t)^2}$

find the velocity and acceleration at the end of 1 second, and at the end of 16 seconds. When $t = 1$, $v = -2$ and $\alpha = \frac{2}{5}$.

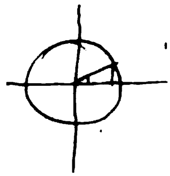
$t = 16$, $v = -\frac{1}{2}$, $\alpha = \frac{1}{40}$

3. If a point moves in a fixed path so that

$$s = \sqrt{t},$$

show that the acceleration is negative and proportional to the cube of the velocity. Find the value of the acceleration at the end of one second, and at the end of nine seconds. $-\frac{1}{4}$, and $-\frac{1}{108}$.

√4. If a point move in a straight line so that



$$x = a \cos \tfrac{1}{2}\pi t,$$

show that $\alpha = -\tfrac{1}{4}\pi^2 x.$ ✓

5. If $x = a\,\epsilon^t + b\,\epsilon^{-t},$

prove that $\alpha = x.$

6. If a point referred to rectangular coordinate axes move so that

$$x = a \cos t + b \qquad \text{and} \qquad y = a \sin t + c,$$

show that its velocity will be uniform. Find the equation of the path described.

Eliminate t from the given equations.

7. A projectile moves in the parabola whose equation is

$$y = x \tan \alpha - \frac{g}{2\,V^2 \cos^2 \alpha}\,x^2,$$

(the axis of y being vertical) with a uniform horizontal velocity

$$v_x = V \cos \alpha\,;$$

find the velocity in the curve, and the vertical acceleration.

$v = \sqrt{(V^2 - 2gy)}$, and $\alpha_y = -g$.

8. A point moves in the curve, whose equation is

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}},$$

so that v_x is constant and equal to k; find the acceleration in the direction of the axis of y.

$$\alpha_y = \frac{a^{\frac{2}{3}} k^2}{3x^{\frac{4}{3}} y^{\frac{1}{3}}}.$$

9. If a point move so that $v = \sqrt{(2gx)}$; determine the acceleration. *Use equation* (1), *Art.* 71. $\alpha = g$.

10. If a point move so that we have

$$v^2 = c - \mu \log x,$$

determine the acceleration. $\alpha = -\dfrac{\mu}{2x}$.

11. If a point move so that we have

$$v^2 = c + \frac{2\mu}{\sqrt{(x^2 + b^2)}},$$

determine the acceleration. $\alpha = -\dfrac{\mu x}{(x^2 + b^2)^{\frac{3}{2}}}$.

12. The velocity of a point is inversely proportional to the square of its distance from a fixed point of the straight line in which it moves, the velocity being 2 feet per second when the distance is six inches; determine the acceleration at a given distance s from the fixed point.

$$-\frac{1}{2s^5} \text{ feet.}$$

13. The velocity of a point moving in a straight line is m times its distance from a fixed point at the perpendicular distance a from the straight line; determine the acceleration at the distance x from the foot of the perpendicular. $\alpha = m^2 x$.

14. The relation between x and t being expressed by

$$t\sqrt{\frac{2\mu}{a}} = \sqrt{(ax - x^2)} - \tfrac{1}{2} a \operatorname{vers}^{-1} \frac{2x}{a};$$

find the acceleration in terms of x. $\alpha = -\dfrac{\mu}{x^2}$.

15. A point moves in the hyperbola

$$y^2 = p^2 x^2 + q^2$$

in such a manner that v_x has the constant value c; prove that

$$v_y^2 = p^2c^2 - \frac{p^2c^2q^2}{y^2},$$

and thence derive a_y by equation (1), Art. 71.

$$a_y = \frac{p^2c^2q^2}{y^3}.$$

16. A point describes the conic section

$$y^2 = 2mx + nx^2,$$

v_x having the constant value c; determine the value of a_y.

Express v_y^2 in terms of y, and proceed as in Example 15.

$$a_y = -\frac{m^2c^2}{y^3}.$$

XI.

Successive Derivatives.

74. The derivative of $f(x)$ is another function of x, which we have denoted by $f'(x)$; if we take the derivative of the latter, we obtain still another function of x, which is called the second derivative of the original function $f(x)$, and is denoted by $f''(x)$. Thus if

$$f(x) = x^3, \qquad f'(x) = 3x^2, \qquad \text{and} \qquad f''(x) = 6x.$$

Similarly the derivative of $f''(x)$ is denoted by $f'''(x)$, and is called the third derivative of $f(x)$; etc. When one of these successive derivatives has a constant value, the next and all succeeding derivatives evidently vanish. Thus, in the above example, $f'''(x) = 6$, consequently, in this case, $f^{\text{iv}}(x)$ and all higher derivatives vanish.

The Geometrical Meaning of the Second Derivative.

75. If the curve whose equation is

$$y = f(x)$$

be constructed, we have seen (Art. 26) that

$$\frac{dy}{dx} = f'(x) = \tan\phi,$$

ϕ being the inclination of the curve to the axis of x; hence

$$f''(x) = \frac{d(\tan\phi)}{dx}.$$

If now the value of this derivative be *positive*, $\tan\phi$ will be an *increasing* function of x, as in Fig. 6, in which, as we proceed toward the right, $\tan\phi$ (at first negative) increases algebraically throughout. In this case, therefore, the curve appears *concave when viewed from above*.

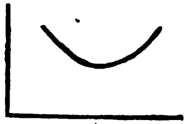


FIG. 6.

On the other hand, if $f''(x)$ be *negative*, $\tan\phi$ will be a decreasing function of x, as in Fig. 7, in which, as we proceed toward the right, $\tan\phi$ decreases algebraically throughout, the curve appearing *convex when viewed from above*.

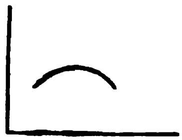


FIG. 7.

76. A point which separates a concave from a convex portion of a curve is called a *point of inflexion*, or a *point of contrary flexure*.

It is obvious from the preceding article that, at a point of inflexion, like P in Fig. 8, $f''(x)$ must *change sign;* hence at such a point, the value of this derivative must become either zero or infinity.

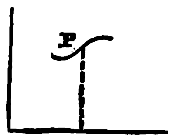


FIG. 8.

77. When a curve is described by a moving point, the character of the curvature is dependent upon the component accelerations of the motion. For, if we put

$$v_x = c, \qquad \text{or} \qquad dx = c\,dt,$$

c denoting a constant, we have

$$f'(x) = \frac{dy}{c\,dt};$$

and hence $$f''(x) = \frac{1}{c^2} \cdot \frac{d^2y}{dt^2} = \frac{\alpha_y}{c^2}.$$

Whence it follows that, if v_x is constant, α_y and $f''(x)$ have the same sign, and consequently that a portion of a curve which is concave when viewed from above is one in which α_y is positive when α_x is zero.

Successive Differentials.

78. The successive differentials of a *function* of x involve the successive differentials of x; thus, if

$$y = x^3,$$

we have $$dy = 3x^2 dx,$$

$$d^2y = 6x\,(dx)^2 + 3x^2\,d^2x,$$

and $$d^3y = 6(dx)^3 + 18x\,dx\,d^2x + 3x^2\,d^3x.$$

In general, if

$$y = f(x),$$

$$dy = f'(x)\,dx,$$

$$d^2y = f''(x)\,(dx)^2 + f'(x)\,d^2x,$$

and $$d^3y = f'''(x)\,(dx)^3 + 3f''(x)\,dx\,d^2x + f'(x)\,d^3x.$$

Equicrescent Variables.

79. A variable is said to be *equicrescent* when its rate is constant; since dt in the expression $\dfrac{dx}{dt}$ is assumed to be constant, dx is also constant, when x is equicrescent.

In expressing the differentials of a function, it is admissible

to assume the independent variable to be equicrescent, since the differential of this variable is arbitrary. This hypothesis greatly simplifies the expressions for the second and higher differentials of functions of x, inasmuch as it is evidently equivalent to making all differentials of x higher than the first vanish. Thus, in the general expressions for d^2y and d^3y given in the preceding article, all the terms except the first disappear, and it is easy to see that, in general, we shall have

$$d^ny = f^n(x)\,(dx)^n,$$

when x is equicrescent.

80. From the above equation we derive

$$\frac{d^ny}{dx^n} = f^n(x).$$

The expression in the first member of this equation is the usual symbol for the nth derivative of y regarded as a function of x. The nth *differential* which occurs in this symbol is always understood to denote the value which this differential assumes *when the variable indicated in the denominator is equicrescent.*

The symbol $\dfrac{d}{dx}$ is frequently used to denote the operation of taking the derivative with reference to x, and similarly the symbol $\left(\dfrac{d}{dx}\right)^n$, or $\dfrac{d^n}{dx^n}$, is used to denote the operation of taking the derivative with respect to x, n times in succession.

Implicit Functions.

81. When y is given as an implicit function of x, the higher derivatives, like the first derivative (Art. 66), can in general be found only in terms of x and y; hence the numerical values of these derivatives can be determined only for known simulta-

neous values of x and y. The following examples will serve to illustrate the method of finding such derivatives.

Given
$$\log(x+y) = x - y; \quad \ldots \ldots \quad (1)$$

we obtain, by differentiating and reducing,

$$(x+y+1)\,dy + (1-x-y)\,dx = 0; \quad \ldots \quad (2)$$

whence
$$\frac{dy}{dx} = \frac{x+y-1}{x+y+1}. \quad \ldots \ldots \quad (3)$$

Differentiating and dividing by dx,

$$\frac{d^2y}{dx^2} = \frac{(x+y+1)\left(1+\frac{dy}{dx}\right) - (x+y-1)\left(1+\frac{dy}{dx}\right)}{(x+y+1)^2};$$

substituting the value of $\frac{dy}{dx}$, we obtain

$$\frac{d^2y}{dx^2} = \frac{4(x+y)}{(x+y+1)^3}. \quad \ldots \ldots \quad (4)$$

In like manner, the third derivative may be found.

Simultaneous values of x and y are readily found in this case. Thus, if we put $x+y=1$, we have $x-y=0$, whence $x=\frac{1}{2}$ and $y=\frac{1}{2}$; by substituting these values in (3) and (4) we obtain

$$\left.\frac{dy}{dx}\right]_{\frac{1}{2},\frac{1}{2}} = 0, \quad \text{and} \quad \left.\frac{d^2y}{dx^2}\right]_{\frac{1}{2},\frac{1}{2}} = \frac{1}{2}.$$

Differential Equations.

82. An equation of which each term contains a first differential is called a *differential equation of the first order.* Thus, equation (2) of the preceding article is a differential equation

of the first order between x and y. Such an equation is obviously equivalent to a relation between x, y, and the first derivative of y with respect to x (or of x with respect to y).

By differentiating equation (2), Art. 81, we obtain

$$(1 + x + y)\, d^2y + (1 - x - y)\, d^2x + (dy)^2 - (dx)^2 = 0. \quad . \quad . \quad (5)$$

It is obvious that each term of such an equation must contain either a second differential or the product of two first differentials; an equation of this character is called *a differential equation of the second order.*

If x and y are functions of a third variable t, equation (5) may be converted into a relation between x, y, and their derivatives with respect to t, by dividing each term by $(dt)^2$. When one of the variables (x or y) is regarded as a function of the other, the independent variable is made equicrescent. Thus, if x be regarded as the independent variable, equation (5) becomes

$$(1 + x + y)\, d^2y + (dy)^2 - (dx)^2 = 0,$$

or

$$(1 + x + y)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 - 1 = 0,$$

a relation between x, y, and the derivatives of y with reference to x.

83. A given equation and the differential equations derived from it are sometimes so combined as to produce a differential equation having certain desired characteristics. Thus, if it be required to derive from

$$y = \sin^{-1} x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

a differential equation free from transcendental functions and radicals, we have

$$\frac{dy}{dx} = \frac{1}{\sqrt{(1 - x^2)}};$$

whence $$\frac{d^2y}{dx^2} = \frac{x}{(1-x^2)^{\frac{3}{2}}} = \frac{x}{1-x^2}\cdot\frac{dy}{dx},$$

or $$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0,$$

a differential equation of the required form.

Examples XI.

1. Find the second derivative of $\sec x$, and distinguish the concave from the convex portions of the curve $y = \sec x$. Also show that the curve $y = \log x$ is everywhere convex.

✓ 2. Find the points of inflexion in the curve $y = \sin x$. $x = n\pi$

✓ 3. Find the point of inflexion of the curve

$$y = 2x^3 - 3x^2 - 12x + 6.$$

The point is $(\frac{1}{2}, -\frac{1}{2})$. ✓

4. Show that the curve $y = \tan x$ is concave when y is positive, and convex when y is negative.

5. Find the points of inflexion of the curve

$$y = x^4 - 2x^3 - 12x^2 + 11x + 24.$$

The points are $(2, -2)$ and $(-1, 4)$.

6. If $f(x) = \dfrac{1+x}{1-x}$, find $f^{\text{v}}(x)$. $\qquad f^{\text{v}}(x) = \dfrac{240}{(1-x)^6}$. ✓

7. If $f(x) = \dfrac{a}{x^n}$, find $f'''(x)$. $\qquad f'''(x) = -\dfrac{n(n+1)(n+2)a}{x^{n+3}}$. ✓

8. If y is a function of x of the form

$$Ax^n + Bx^{n-1} + \cdots + Mx + N,$$

prove that $$\frac{d^ny}{dx^n} = 1.2.3\cdots nA. \checkmark$$

9. If $f(x) = b^{ax}$, find $f^{\text{v}}(x)$. $\qquad f^{\text{v}}(x) = a^5 (\log b)^5\, b^{ax}$.

10. If $f(x) = x^3 \log (mx)$, find $f^{\text{iv}}(x)$. $\qquad f^{\text{iv}}(x) = \dfrac{6}{x}$.

11. If $f(x) = \log \sin x$, find $f'''(x)$. $\qquad f'''(x) = \dfrac{2 \cos x}{\sin^3 x}$.

12. If $f(x) = \sec x$, find $f''(x)$ and $f'''(x)$.

$f''(x) = 2 \sec^3 x - \sec x$, and $f'''(x) = \sec x \tan x\, (6 \sec^2 x - 1)$.

13. If $f(x) = \tan x$, find $f'''(x)$ and $f^{\text{iv}}(x)$.

$f'''(x) = 6 \sec^4 x - 4 \sec^2 x$, and $f^{\text{iv}}(x) = 8 \tan x \sec^2 x\, (3 \sec^2 x - 1)$.

14. If $f(x) = x^x$, find $f''(x)$. $\qquad f''(x) = x^x (1 + \log x)^2 + x^{x-1}$.

15. If $y = \varepsilon^{\frac{1}{x}}$, find $\dfrac{d^3y}{dx^3}$. $\qquad \dfrac{d^3y}{dx^3} = -\dfrac{1}{x^6}(1 + 6x + 6x^2)\, \varepsilon^{\frac{1}{x}}$.

16. If $y = \varepsilon^{-x^2}$, find $\dfrac{d^3y}{dx^3}$. $\qquad \dfrac{d^3y}{dx^3} = 4x\,(3 - 2x^2)\, \varepsilon^{-x^2}$.

17. If $y = \log (\varepsilon^x + \varepsilon^{-x})$, find $\dfrac{d^3y}{dx^3}$. $\qquad \dfrac{d^3y}{dx^3} = -8\, \dfrac{\varepsilon^x - \varepsilon^{-x}}{(\varepsilon^x + \varepsilon^{-x})^3}$.

18. If $y = \dfrac{1}{\varepsilon^x - 1}$, find $\dfrac{d^2y}{dx^2}$ and $\dfrac{d^4y}{dx^4}$.

$$\frac{d^2y}{dx^2} = \frac{\varepsilon^{2x} + \varepsilon^x}{(\varepsilon^x - 1)^3}, \text{ and } \frac{d^4y}{dx^4} = \frac{\varepsilon^x + 11\varepsilon^{2x} + 11\varepsilon^{3x} + \varepsilon^{4x}}{(\varepsilon^x - 1)^5}.$$

19. If $y = \sin^{-1} x$, find $\dfrac{d^4y}{dx^4}$. $\qquad \dfrac{d^4y}{dx^4} = \dfrac{9x + 6x^3}{(1 - x^2)^{\frac{7}{2}}}$.

20. If $y = \varepsilon^{\sin x}$, find $\dfrac{d^3y}{dx^3}$.

$$\frac{d^3y}{dx^3} = -\varepsilon^{\sin x} \cos x \sin x\, (\sin x + 3).$$

21. If $y = \dfrac{x}{1 + \log x}$, find $\dfrac{d^2y}{dx^2}$. $\qquad \dfrac{d^2y}{dx^2} = \dfrac{1 - \log x}{x(1 + \log x)^3}$.

22. Find the value of $d^3(\varepsilon^x)$, when x is not equicrescent.

$$d^3(\varepsilon^x) = \varepsilon^x(dx)^3 + 3\varepsilon^x\, d^2x\, dx + \varepsilon^x\, d^3x.$$

23. Find the value of $\dfrac{d^3}{dt^3}(\sin\theta)$, θ being a function of t.

$$\frac{d^3}{dt^3}(\sin\theta) = -\cos\theta\left(\frac{d\theta}{dt}\right)^3 - 3\sin\theta\,\frac{d\theta}{dt}\cdot\frac{d^2\theta}{dt^2} + \cos\theta\,\frac{d^3\theta}{dt^3}.$$

24. If y is a function of x, and if $u = y^2\log y$; find $\dfrac{d^2u}{dx^2}$.

$$\frac{d^2u}{dx^2} = (2\log y + 3)\left(\frac{dy}{dx}\right)^2 + y(2\log y + 1)\frac{d^2y}{dx^2}.$$

25. If $u = \dfrac{x}{y}$, y being a function of x, find $\dfrac{d^2u}{dx^2}$.

$$\frac{d^2u}{dx^2} = -\frac{2}{y^2}\frac{dy}{dx} + \frac{2x}{y^3}\left(\frac{dy}{dx}\right)^2 - \frac{x}{y^2}\frac{d^2y}{dx^2}.$$

26. If $y - 1 - x\varepsilon^y = 0$, find $\dfrac{d^2y}{dx^2}$. $\qquad \dfrac{d^2y}{dx^2} = \dfrac{3 - y}{(2 - y)^3}\cdot\varepsilon^{2y}$.

27. If $y = \tan(x + y)$, find $\dfrac{d^3y}{dx^3}$. $\qquad \dfrac{d^3y}{dx^3} = -\dfrac{2(5 + 8y^2 + 3y^4)}{y^8}$.

28. If $y^2 + y = x^2$, find $\dfrac{d^3y}{dx^3}$. $\qquad \dfrac{d^3y}{dx^3} = -\dfrac{24x}{(1 + 2y)^5}$.

29. Given $\varepsilon^x + x = \varepsilon^y + y$, to find $\dfrac{d^2y}{dx^2}$.

$$\frac{d^2y}{dx^2} = \frac{(\varepsilon^{x+y} - 1)(x - y)}{(\varepsilon^y + 1)^3}.$$

30. Given $\varepsilon^y + xy - \varepsilon = 0$, to find $\dfrac{d^2y}{dx^2}$.

$$\frac{d^2y}{dx^2} = y\cdot\frac{(2 - y)\varepsilon^y + 2x}{(\varepsilon^y + x)^3}.$$

31. Given $y^3 - 3axy + x^3 = 0$, to find $\dfrac{d^2y}{dx^2}$.

$$\frac{d^2y}{dx^2} = -\frac{2a^3xy}{(y^2 - ax)^3}$$

32. Given $xy = a\varepsilon^x + b\varepsilon^{-x}$, to deduce the following differential equation :—

$$x\frac{d^2y}{dx} + 2\frac{dy}{dx} - xy = 0.$$

33. Given $y = \varepsilon^x \sin x$, to derive

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0.$$

34. Given $y = a\sin x + b\cos x$, to derive

$$\frac{d^2y}{dx^2} + y = 0.$$

35. Given $y = a\cos\log x + b\sin\log x$, to derive

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0.$$

36. Given $y = (\sin^{-1}x)^2$, to derive

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2.$$

37. Given $y = a\varepsilon^x + b\varepsilon^{-x} + c\sin(x + m)$, to derive

$$\frac{d^4y}{dx^4} - y = 0.$$

38. Given $y = \dfrac{\varepsilon^x + \varepsilon^{-x}}{\varepsilon^x - \varepsilon^{-x}}$, to derive

$$\frac{dy}{dx} = 1 - y^2.$$

39. Given $y = \log[x + \sqrt{(a^2 + x^2)}]$, to derive

$$(a^2 + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0.$$

40. Given $y = [x + \sqrt{(x^2 + 1)}]^m$, to derive

$$(x^2 + 1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - m^2y = 0.$$

41. Given the equation of the circle

$$(x - a)^2 + (y - b)^2 = c^2,$$

to derive a differential equation independent of a and b.

$$c\frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}.$$

XII.

Expressions for the n*th Derivative.*

84. The expression for the nth derivative is readily found in the case of certain functions. Thus, if

$$f(x) = \log x,$$

$$f'(x) = \frac{1}{x} = x^{-1},$$

$$f''(x) = -x^{-2},$$

$$f'''(x) = 1\cdot 2\, x^{-3},$$

and
$$f^{\text{iv}}(x) = -1\cdot 2\cdot 3\, x^{-4};$$

whence, it is evident that

$$f^n(x) = (-1)^{n-1}\, 1\cdot 2 \cdots (n-1)\, x^{-n}. \quad . \quad . \quad . \quad . \quad (1)$$

Again, if $\quad f(x) = \sin x,$

$$f'(x) = \cos x = \sin(x + \tfrac{1}{2}\pi),$$

$$f''(x) = \sin(x + \pi),$$

$$f'''(x) = \sin(x + \tfrac{3}{2}\pi),$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

therefore $$f^n(x) = \sin\left(x + \frac{n}{2}\pi\right). \quad . \quad . \quad . \quad . \quad . \quad (2)$$

85. Artifices of a less obvious character are sometimes necessary in expressing the nth derivatives of functions. Let

$$y = \varepsilon^{ax} \cos(bx), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

then $$dy = \varepsilon^{ax}[a \cos(bx) - b \sin(bx)]\, dx.$$

Employing an auxiliary constant α determined by

$$b = a \tan \alpha, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

we have

$$\frac{dy}{dx} = \frac{a}{\cos \alpha} \varepsilon^{ax}[\cos(bx) \cos \alpha - \sin(bx) \sin \alpha],$$

$$= \frac{a}{\cos \alpha} \varepsilon^{ax} \cos(bx + \alpha).$$

In like manner, we obtain

$$\frac{d^2y}{dx^2} = \frac{a^2}{\cos^2 \alpha} \varepsilon^{ax} \cos(bx + 2\alpha),$$

and in general $$\frac{d^ny}{dx^n} = \frac{a^n}{\cos^n \alpha} \varepsilon^{ax} \cos(bx + n\alpha),$$

or, since, by equation (2),

$$\cos \alpha = \frac{a}{\sqrt{(a^2 + b^2)}},$$

$$\frac{d^ny}{dx^n} = (a^2 + b^2)^{\frac{n}{2}} \varepsilon^{ax} \cos\left(bx + n \tan^{-1}\frac{b}{a}\right). \quad . \quad . \quad (3)$$

86. To find the nth derivative of the function

$$y = \frac{1}{a^2 + x^2}, \quad \ldots \ldots \ldots \quad (1)$$

we put

$$x = a \cot \theta, \quad \ldots \ldots \ldots \quad (2)$$

θ being an auxiliary variable, whence

$$y = \frac{1}{a^2} \sin^2\theta,$$

and

$$\frac{dy}{dx} = \frac{1}{a^2}\, 2 \sin\theta \cos\theta \frac{d\theta}{dx}. \quad \ldots \ldots \quad (3)$$

Differentiating equation (2), we have

$$dx = -a \operatorname{cosec}^2\theta\, d\theta,$$

or

$$\frac{d\theta}{dx} = -\frac{\sin^2\theta}{a};$$

substituting in equation (3),

$$\frac{dy}{dx} = -\frac{1}{a^3} \sin 2\theta \sin^2\theta,$$

whence

$$\frac{d^2y}{dx^2} = -\frac{1}{a^3} \cdot 2 \sin\theta (\sin 2\theta \cos\theta + \cos 2\theta \sin\theta) \frac{d\theta}{dx},$$

$$= \frac{1 \cdot 2}{a^4} \sin 3\theta \sin^3\theta.$$

In a similar manner, we obtain

$$\frac{d^3y}{dx^3} = -\frac{1 \cdot 2 \cdot 3}{a^5} \sin 4\theta \sin^4\theta,$$

and in general

$$\frac{d^ny}{dx} = (-1)^n \frac{1 \cdot 2 \cdot 3 \cdots n}{a^{n+2}} \sin\left[(n+1)\theta\right] (\sin\theta)^{n+1}. \quad \ldots \quad (4)$$

Since we have, from equation (2),

$$\sin\theta = \frac{a}{\sqrt{(a^2+x^2)}},$$

equation (4) may be written thus—

$$\frac{d^n}{dx^n}\left(\frac{1}{a^2+x^2}\right) = (-1)^n \frac{1\cdot 2\cdot 3\cdots n}{a(a^2+x^2)^{\frac{n+1}{2}}}\sin\left[(n+1)\tan^{-1}\frac{a}{x}\right]. \quad . \quad . \quad (5)$$

87. Since $\qquad \dfrac{d}{dx}\tan^{-1}\dfrac{x}{a} = \dfrac{a}{a^2+x^2},$

we have $\qquad \dfrac{d^n}{dx^n}\tan^{-1}\dfrac{x}{a} = a\dfrac{d^{n-1}}{dx^{n-1}}\left(\dfrac{1}{a^2+x^2}\right).$

Hence we obtain from equation (5)

$$\frac{d^n}{dx^n}\tan^{-1}\frac{x}{a} = (-1)^{n-1}\frac{1\cdot 2\cdot 3\cdots(n-1)}{(a^2+x^2)^{\frac{n}{2}}}\sin\left[n\tan^{-1}\frac{a}{x}\right].$$

Leibnitz' Theorem.

88. We proceed to deduce an expression for the nth differential of the product of two variables in terms of the successive derivatives of these variables.

From $\qquad d(uv) = udv + vdu,$

we derive, by successive differentiation,

$$d^2(uv) = u\,d^2v + 2du\,dv + v\,d^2u,$$

and $\qquad d^3(uv) = u\,d^3v + 3du\,d^2v + 3d^2u\,dv + v\,d^3u;$

from which the law of the indices in the expansion of $d^n(uv)$ is readily inferred. We also see that, when $n = 2$ and when $n = 3$, the coefficients are identical with those in the expansion of

$(a+b)^n$. That this is true for all integral values of n may be shown in the following manner. Assuming that

$$d^n(uv) = u\,d^nv + n\,du\,d^{n-1}v + \frac{n(n-1)}{2}\,d^2u\,dv^{n-2} + \cdots, \quad . . \ (1)$$

we have, by differentiating,

$$d^{n+1}(uv) = ud^{n+1}v + (n+1)du\,d^nv + n\left[1 + \frac{n-1}{2}\right]d^2u\,d^{n-1}v$$
$$+ n\frac{n-1}{2}\left[1 + \frac{n-2}{3}\right]d^3u\,d^{n-2}v + \cdots,$$

or, reducing,

$$d^{n+1}(uv) = u\,d^{n+1}v + (n+1)\,du\,d^nv + \frac{(n+1)n}{2}\,d^2u\,d^{n-1}v + \cdots,$$

in which the coefficients follow the same law. Hence, if equation (1) is true for any value of n, it is true for all greater values of n; now equation (1) is true when $n = 1$, therefore it is true universally.

The result expressed in equation (1) is called from the name of its discoverer the *Theorem of Leibnitz*.

When u and v are functions of the same independent variable we may, by dividing by $(dx)^n$, convert the differentials in equation (1) into derivatives.

89. By means of this theorem the nth derivative of the function

$$\frac{x}{a^2+x^2}$$

may now be deduced, since the nth derivative of $\frac{1}{a^2+x^2}$ has already been obtained in Art. 86. Putting $u = x$ in equation (1), $\frac{du}{dx} = 1$, and all higher derivatives of u vanish; therefore we have

$$\frac{d^n(xv)}{dx^n} = x\frac{d^nv}{dx^n} + n\frac{d^{n-1}v}{dx^{n-1}}.$$

Putting $v = \frac{1}{a^2 + x^2}$ in this result, and employing the value of $\frac{d^n}{dx^n}\left(\frac{1}{a^2 + x^2}\right)$ given in equation (4), Art. 86,

$$\frac{d^n}{dx^n}\left(\frac{x}{a^2 + x^2}\right) = (-1)^n \frac{1 \cdot 2 \cdots n}{a^{n+1}}\left[\frac{x}{a}\sin\left[(n+1)\theta\right](\sin\theta)^{n+1} - \sin(n\theta)(\sin\theta)^n\right],$$

and, since $x = a\cot\theta$, [equation (2) Art. 86]

$$\frac{d^n}{dx^n}\left(\frac{x}{a^2 + x^2}\right) = (-1)^n \frac{1 \cdot 2 \cdots n}{a^{n+1}}(\sin\theta)^n\left[\sin\left[(n+1)\theta\right]\cos\theta - \sin(n\theta)\right],$$

or, since $\sin(n\theta) = \sin\left[(n+1)\theta\right]\cos\theta - \sin\theta\cos\left[(n+1)\theta\right]$

$$\frac{d^n}{dx^n}\left(\frac{x}{a^2 + x^2}\right) = (-1)^n \frac{1 \cdot 2 \cdots n}{a^{n+1}}(\sin\theta)^{n+1}\cos\left[(n+1)\theta\right].$$

90. An important result is obtained by applying Leibnitz' theorem to the function

$$u\,\varepsilon^{ax},$$

u denoting any function of x.

We obviously have

$$\frac{d^n}{dx^n}(\varepsilon^{ax}) = a^n \varepsilon^{ax};$$

whence putting $v = \varepsilon^{ax}$ in equation (1), Art. 88,

$$\frac{d^n}{dx^n}\left(u\,\varepsilon^{ax}\right) = \varepsilon^{ax}\left[a^n u + n a^{n-1}\frac{du}{dx} + \frac{n(n-1)}{1 \cdot 2}a^{n-2}\frac{d^2u}{dx^2} + \cdots + \frac{d^n u}{dx^n}\right]$$

This result is frequently written in the symbolic form

$$\frac{d^n}{dx^n}\left(u\varepsilon^{ax}\right) = \varepsilon^{ax}\left(a + \frac{d}{dx}\right)^n u,$$

which is to be understood as indicating that $\left(a + \frac{d}{dx}\right)^n$ is to be expanded as if it were an ordinary binomial, and that u is to be affixed to each term of the expansion.

The Numerical Values of Successive Derivatives for Particular Values of the Independent Variable.

91. The theorem of Leibnitz is sometimes employed in deducing a general relation between two or more successive derivatives of a function, by means of which their numerical values may be computed for given values of x. Thus, if

$$y = \sin(m \sin^{-1} x), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

we have
$$\frac{dy}{dx} = \frac{m \cos[m \sin^{-1} x]}{\sqrt{(1 - x^2)}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and

$$\frac{d^2y}{dx^2} = \frac{-m^2 \sin[m \sin^{-1} x] + m \cos[m \sin^{-1} x] \dfrac{x}{\sqrt{(1 - x^2)}}}{1 - x^2}. \quad . \quad (3)$$

Whence,
$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2 y = 0.$$

Taking the nth derivative of each term by means of Leibnitz, theorem, we find

$$\left.\begin{aligned} (1 - x^2)\frac{d^{n+2}y}{dx^{n+2}} - 2n\,x\frac{d^{n+1}y}{dx^{n+1}} - n(n-1)\frac{d^ny}{dx^n} \\ - x\frac{d^{n+1}y}{dx^{n+1}} - n\frac{d^ny}{dx^n} \\ + m^2\frac{d^ny}{dx^n} \end{aligned}\right\} = 0,$$

or

$$(1 - x^2)\frac{d^{n+2}y}{dx^{n+2}} - (2n + 1)x\frac{d^{n+1}y}{dx^{n+1}} + (m^2 - n^2)\frac{d^ny}{dx^n} = 0, \quad . \; . \; . \; (4)$$

the general relation required.

92. Equation (4) may be employed to compute the numerical values of the derivatives of y corresponding to $x = 0$. Substituting this value of x, equation (4) becomes

$$\left.\frac{d^{n+2}y}{dx^{n+2}}\right]_0 = (n^2 - m^2)\left.\frac{d^ny}{dx^n}\right]_0.$$

Now from (2) and (3), we obtain

$$\left.\frac{dy}{dx}\right]_0 = m, \qquad \text{and} \qquad \left.\frac{d^2y}{dx^2}\right]_0 = 0;$$

hence when n is even $\quad \left.\dfrac{d^ny}{dx^n}\right]_0 = 0,$

and, putting n equal to 1, 3, 5, etc., we have

$$\left.\frac{d^3y}{dx^3}\right]_0 = m(1 - m^2), \qquad \left.\frac{d^5y}{dx^5}\right]_0 = m(1 - m^2)(9 - m^2), \text{ etc.}$$

Examples XII.

1. $y = x^m$; find $\dfrac{d^ny}{dx^n}$. $\quad \dfrac{d^ny}{dx^n} = m(m - 1)\cdots(m - n + 1)x^{m-n}$.

2. $y = \dfrac{1}{(a - x)^m}$.

$$\frac{d^ny}{dx^n} = m(m + 1)\cdots(m + n - 1)(a - x)^{-m-n}.$$

3. $y = \cos mx$. $\quad \dfrac{d^ny}{dx^n} = m^n \cos\left(mx + \dfrac{n}{2}\pi\right)$.

4. $y = \log_b(a+x)$. $\qquad \dfrac{d^n y}{dx^n} = (-1)^{n-1}\dfrac{1\cdot 2\cdots(n-1)}{\log b\,(a+x)^n}.$

5. $y = \log(1-mx)$.

$$\frac{d^n y}{dx^n} = -1\cdot 2\cdots(n-1)m^n(1-mx)^{-n}.$$

6. $y = \varepsilon^{ax}\sin bx$. $\qquad \dfrac{d^n y}{dx^n} = (a^2+b^2)^{\frac{n}{2}}\varepsilon^{ax}\sin\left[bx + n\tan^{-1}\dfrac{b}{a}\right].$

7. $y = x^m \log x$, prove that $\dfrac{d^{m+1}y}{dx^{m+1}} = \dfrac{m(m-1)\cdots 1}{x}.$

8. $y = \varepsilon^{x\cos\alpha}\cos(x\sin\alpha)$. $\qquad \dfrac{d^n y}{dx^n} = \varepsilon^{x\cos\alpha}\cos(x\sin\alpha + n\alpha).$

9. $y = \cos^2 x$. $\qquad \dfrac{d^n y}{dx^n} = 2^{n-1}\cos\left(2x + \dfrac{n}{2}\pi\right).$

10. $y = \dfrac{1}{a^2-x^2} = \dfrac{1}{2a}\left[\dfrac{1}{a-x} + \dfrac{1}{a+x}\right].$

$$\frac{d^n y}{dx^n} = \frac{1\cdot 2\cdots n}{2a}\left[\frac{1}{(a-x)^{n+1}} + \frac{(-1)^n}{(a+x)^{n+1}}\right].$$

11. $y = \dfrac{x}{a^2-x^2} = \frac{1}{2}\left[\dfrac{1}{a-x} - \dfrac{1}{a+x}\right].$

$$\frac{d^n y}{dx^n} = \frac{1\cdot 2\cdots n}{2}\left[\frac{1}{(a-x)^{n+1}} - \frac{(-1)^n}{(a+x)^{n+1}}\right].$$

12. $y = x\varepsilon^{2x}$; find $\dfrac{d^n y}{dx^n}$, by Leibnitz' theorem.

$$\frac{d^n y}{dx^n} = 2^{n-1}(n+2x)\varepsilon^{2x}.$$

13. $y = x^2\varepsilon^x$. $\qquad \dfrac{d^n y}{dx^n} = [n(n-1) + 2nx + x^2]\varepsilon^x.$

14. $y = x^3 v$, v being a function of x.

$$\frac{d^n y}{dx^n} = x^3 \frac{d^n v}{dx^n} + 3nx^2 \frac{d^{n-1} v}{dx^{n-1}} + 3n(n-1)\, x \frac{d^{n-2} v}{dx^{n-2}}$$
$$+ n(n-1)(n-2) \frac{d^{n-3} v}{dx^{n-3}}.$$

15. $y = x^3 \log x$. When $n > 3$,

$$\frac{d^n y}{dx^n} = (-1)^{n-1}\, 1 \cdot 2 \;.\;.\;.\; n \cdot x^{-n+3} \left[\frac{1}{n} - \frac{3}{n-1} + \frac{3}{n-2} - \frac{1}{n-3}\right].$$

16. $y = x \sin x$. $\qquad \dfrac{d^n y}{dx^n} = x \sin\left(x + n\dfrac{\pi}{2}\right) - n \cos\left(x + n\dfrac{\pi}{2}\right)$.

17. $y = (1 - x)^n x^n$.

$$\frac{d^n y}{dx^n} = 1 \cdot 2 \cdots n \left\{ (1-x)^n - n^2 x (1-x)^{n-1} \right.$$
$$\left. + \left[\frac{n(n-1)}{1 \cdot 2}\right]^2 x^2 (1-x)^{n-2} - \cdots \right\}.$$

18. If $y = x^m \epsilon^{ax}$, prove that

$$\frac{d^n y}{dx^n} = \epsilon^{ax}\left[a^n x^m + n a^{n-1} m x^{m-1}\right.$$
$$\left. + \frac{n(n-1)}{1 \cdot 2} a^{n-2} m(m-1) x^{m-2} + \cdots \right],$$

and thence show that

$$x^{m-n} \frac{d^m}{dx^m}\left(\epsilon^{ax} x^n\right) = a^{m-n} \frac{d^n}{dx^n}\left(\epsilon^{ax} x^m\right).$$

19. If $y = \sin^{-1} x$, derive, by applying Leibnitz' theorem to the differential equation obtained from this function in Art. 83, the result,

$$(1 - x^2) \frac{d^{n+2} y}{dx^{n+2}} - (2n+1) x \frac{d^{n+1} y}{dx^{n+1}} - n^2 \frac{d^n y}{dx^n} = 0,$$

and thence deduce by the method of Art. 92 the values of the successive derivatives when $x = 0$.

$$\left.\frac{d^3y}{dx^3}\right]_0 = 1, \quad \left.\frac{d^5y}{dx^5}\right]_0 = 3^2, \quad \left.\frac{d^7y}{dx^7}\right]_0 = 3^2.5^2, \text{ etc.};$$

and when n is even $\left.\frac{d^ny}{dx^n}\right]_0 = 0.$

20. If $y = \tan^{-1} x$, we have $(1 + x^2)\frac{dy}{dx} - 1 = 0$. Hence derive, by means of Leibnitz' theorem,

$$(1 + x^2)\frac{d^{n+1}y}{dx^{n+1}} + 2nx\frac{d^ny}{dx^n} + n(n-1)\frac{d^{n-1}y}{dx^{n-1}} = 0,$$

and the values of the derivatives when $x = 0$.

$$\left.\frac{dy}{dx}\right]_0 = 1, \quad \left.\frac{d^3y}{dx^3}\right]_0 = -2, \quad \left.\frac{d^5y}{dx^5}\right]_0 = 2\cdot3\cdot4;$$

and when n is even $\left.\frac{d^ny}{dx^n}\right]_0 = 0.$

21. Given $y = \log[x + \sqrt{(a^2 + x^2)}]$, derive

$$(a^2 + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0,$$

and thence $(a^2 + x^2)\frac{d^{n+2}y}{dx^{n+2}} + (2n+1)x\frac{d^{n+1}y}{dx^{n+1}} + n^2\frac{d^ny}{dx^n} = 0;$

also the values of the derivatives when $x = 0$.

$$\left.\frac{dy}{dx}\right]_0 = \frac{1}{a}, \quad \left.\frac{d^3y}{dx^3}\right]_0 = -\frac{1}{a^3}, \quad \left.\frac{d^5y}{dx^5}\right]_0 = \frac{3^2}{a^5};$$

and when n is even $\left.\frac{d^ny}{dx^n}\right]_0 = 0.$

22. Given $y = [x + \sqrt{(a^2 + x^2)}]^m$, derive

$$(a^2 + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - m^2y = 0.$$

and thence $$(a^2 + x^2)\frac{d^{n+2}y}{dx^{n+2}} + (2n + 1)x\frac{d^{n+1}y}{dx^{n+1}} + (n^2 - m^2)\frac{d^n y}{dx^n} = 0;$$

also deduce the values of the derivatives when $x = 0$.

$$\left.\frac{dy}{dx}\right]_0 = ma^{m-1}, \quad \left.\frac{d^2y}{dx^2}\right]_0 = m^2a^{m-2}, \quad \left.\frac{d^3y}{dx^3}\right]_0 = m(m^2 - 1)a^{m-3};$$

$$\left.\frac{d^4y}{dx^4}\right]_0 = m^2(m^2 - 4)a^{m-4}.$$

23. Given $xy = a\epsilon^x + b\epsilon^{-x}$, derive

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = xy,$$

and thence $$x\frac{d^{n+2}y}{dx^{n+2}} + (n + 2)\frac{d^{n+1}y}{dx^{n+1}} - x\frac{d^ny}{dx^n} - n\frac{d^{n-1}y}{dx^{n-1}}.$$

24. Given $y = a\cos\log x + b\sin\log x$, derive

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0,$$

and thence $$\left.\frac{d^{n+2}y}{dx^{n+2}}\right]_1 + (2n + 1)\left.\frac{d^{n+1}y}{dx^{n+1}}\right]_1 + (n^2 + 1)\left.\frac{d^ny}{dx^n}\right]_1 = 0;$$

also the values of the derivatives when $x = 1$.

$$\left.\frac{dy}{dx}\right]_1 = b, \quad \left.\frac{d^2y}{dx^2}\right]_1 = -(a + b), \quad \left.\frac{d^3y}{dx^3}\right]_1 = 3a + b, \text{ and } \left.\frac{d^4y}{dx^4}\right]_1 = -10a.$$

CHAPTER V.

THE EVALUATION OF INDETERMINATE FORMS.

XIII.

Indeterminate or Illusory Forms.

93. WHEN a function is expressed in the form of a fraction each of whose terms is variable, it may happen that, for a certain value of the independent variable, both terms reduce to zero. The function then takes the form $\frac{0}{0}$, and is said to be *indeterminate*, since its value cannot be ascertained by the ordinary process of dividing the value of the numerator by that of the denominator. The function has, nevertheless, a value as determinate for this as for any other value of the independent variable. It is the object of this chapter to show that such definite values exist, and to explain the methods by which they are determined.

may have

The term *illusory form* is often used as synonymous with *indeterminate form*, and these terms are applied indifferently, not only to the form $\frac{0}{0}$, but also to the forms $\frac{\infty}{\infty}$, $\infty \cdot 0$, $\infty - \infty$, and to certain others whose logarithms assume the form $\infty \cdot 0$.

When a function of x takes an illusory form for $x=a$, the corresponding value of the function is sometimes called its ***limiting value*** as x approaches the value a.

94. The values of functions which assume illusory forms may

sometimes be ascertained by making use of certain algebraic transformations. Thus, for example, the function

$$\frac{a - \sqrt{(a^2 - bx)}}{x}$$

takes the form $\frac{0}{0}$ when $x = 0$.

Multiplying both terms by the complementary surd

$$a + \sqrt{(a^2 - bx)},$$

we obtain $$\frac{bx}{x[a + \sqrt{(a^2 - bx)}]} = \frac{b}{a + \sqrt{(a^2 - bx)}}.$$

The last form is not illusory for the given value of x, since the factor which becomes zero has been removed from both terms of the fraction. The value of the fraction for $x = 0$ is evidently $\frac{b}{2a}$.

The following notation is used to indicate this and similar results; viz.,

$$\left.\frac{a - \sqrt{(a^2 - bx)}}{x}\right]_0 = \frac{b}{2a},$$

the subscript denoting that value of the independent variable for which the function is evaluated.

Evaluation by Differentiation.

95. Let $\frac{v}{u}$ represent a function in which both u and v are functions of x, which vanish when $x = a$; in other words, for this value of x, we have $u = 0$, and $v = 0$.

Let P be a moving point of which the abscissa and ordinate are simultaneous values of u and v (x not being represented in the figure); then, denoting the angle POU by θ, and the inclination of the motion of P to the axis of u by ϕ, we have

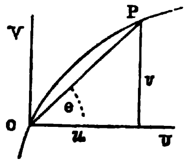


FIG. 9.

$$\tan\theta = \frac{v}{u}, \qquad \text{and} \qquad \tan\phi = \frac{dv}{du}.$$

At the instant when x passes through the value a, u and v being zero by the hypothesis, P passes through the origin; the corresponding value of θ is evidently determined by the direction in which P is moving at that instant, and is therefore equal to the value of ϕ at that point.

Hence the values of $\tan\theta$ and $\tan\phi$ corresponding to $x = a$ are equal, or

$$\left.\frac{v}{u}\right]_{x=a} = \left.\frac{dv}{du}\right]_{x=a};$$

therefore, to determine the value of $\frac{v}{u}$ for $x = a$, we substitute for it the function $\frac{dv}{du}$, whose value is the same as that of the given function, when $x = a$.

96. This result may also be expressed in the following manner: let $f(x)$ and $\phi(x)$ be two functions, such that $f(a) = 0$, and $\phi(a) = 0$; then

$$\frac{f(a)}{\phi(a)} = \frac{f'(a)}{\phi'(a)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

As an illustration, let us take $\frac{\log x}{x - 1}$. When $x = 1$, this function takes the form $\frac{0}{0}$; by the above process, we have

$$\left.\frac{\log x}{x-1}\right]_1 = \left.\frac{x^{-1}}{1}\right]_1 = 1,$$

the required value.

97. Since the substituted function $\dfrac{dv}{du}$ or $\dfrac{f'(x)}{\phi'(x)}$ frequently takes the indeterminate form, several repetitions of the process are sometimes requisite before the value of the function can be ascertained.

For example, the function $\dfrac{1-\cos\theta}{\theta^2}$ takes the form $\dfrac{0}{0}$ when $\theta = 0$; employing the process for evaluating, we have

$$\left.\frac{1-\cos\theta}{\theta^2}\right]_0 = \left.\frac{\sin\theta}{2\theta}\right]_0,$$

which is likewise indeterminate; but, by repeating the process, we obtain

$$\left.\frac{1-\cos\theta}{\theta^2}\right]_0 = \left.\frac{\sin\theta}{2\theta}\right]_0 = \left.\frac{\cos\theta}{2}\right]_0 = \tfrac{1}{2}.$$

98. If the given function, or any of the substituted functions, contains a factor which does not take the indeterminate form, this factor may be evaluated at once, as in the following example.

The function

$$\frac{(1-x)\,\epsilon^x - 1}{\tan^2 x}$$

is indeterminate for $x = 0$. By employing the usual process once, we obtain

$$\left.\frac{(1-x)\,\epsilon^x - 1}{\tan^2 x}\right]_0 = \left.\frac{-x\epsilon^x}{2\sec^2 x\tan x}\right]_0,$$

which is likewise indeterminate; but, before repeating the process, we may evaluate the factor $-\left.\dfrac{\epsilon^x}{2\sec^2 x}\right]_0$. The value of this factor is $-\tfrac{1}{2}$; hence we write

$$\left.\frac{(1-x)\varepsilon^x - 1}{\tan^2 x}\right]_0 = -\left.\frac{x\varepsilon^x}{2\sec^2 x \tan x}\right]_0 = -\tfrac{1}{2}\frac{x}{\tan x}$$

$$= -\tfrac{1}{2}\left.\frac{1}{\sec^2 x}\right]_0 = -\tfrac{1}{2}.$$

99. When the given function can be decomposed into factors each of which takes the indeterminate form, these factors may be evaluated separately. Thus, if the given function be

$$\frac{(\varepsilon^x - 1)\tan^2 x}{x^3},$$

the form
$$\left(\frac{\tan x}{x}\right)^2\left(\frac{\varepsilon^x - 1}{x}\right)$$

may be employed. We have

$$\left.\frac{\tan x}{x}\right]_0 = 1, \quad \text{and} \quad \left.\frac{\varepsilon^x - 1}{x}\right]_0 = 1;$$

hence the value of the given function is unity.

When this method is used, if one of the factors is found to take the value zero while another is infinite, their product, being of the form $0 \cdot \infty$, must be treated by the usual method, since $0 \cdot \infty$ is itself an illusory form.

100. Another mode of decomposing a given function is that of separating it into parts, and substituting the values of such parts as are found on evaluation to be finite.

As an illustration, we take the expression,

$$u_0 = \left.\frac{(\varepsilon^x - \varepsilon^{-x})^2 - 2x^2(\varepsilon^x + \varepsilon^{-x})}{x^4}\right]_0.$$

Each of the fractions into which this function can be decomposed being obviously infinite, we first apply the usual process, thus obtaining

$$u_0 = \frac{2(\varepsilon^x - \varepsilon^{-x})(\varepsilon^x + \varepsilon^{-x}) - 4x(\varepsilon^x + \varepsilon^{-x}) - 2x^2(\varepsilon^x - \varepsilon^{-x})}{4x^3}\Bigg]_0.$$

Separating this expression into two fractions, thus,—

$$u_0 = \frac{(\varepsilon^x + \varepsilon^{-x})(\varepsilon^x - \varepsilon^{-x} - 2x)}{2x^3}\Bigg]_0 - \frac{\varepsilon^x - \varepsilon^{-x}}{2x}\Bigg]_0;$$

the latter is found on evaluation to have a finite value, and the expression reduces to

$$u_0 = \frac{\varepsilon^x - \varepsilon^{-x} - 2x}{x^3}\Bigg]_0 - 1.$$

Hence

$$u_0 = \frac{\varepsilon^x + \varepsilon^{-x} - 2}{3x^2}\Bigg]_0 - 1 = \frac{\varepsilon^x - \varepsilon^{-x}}{6x}\Bigg]_0 - 1 = -\tfrac{2}{3}.$$

Examples XIII.

1. Prove $\dfrac{\sin x}{x}\Big]_0 = 1$, $\dfrac{\tan x}{x}\Big]_0 = 1$, and $\dfrac{\varepsilon^x - 1}{x}\Big]_0 = 1$.

These results are frequently useful in evaluating other functions.
Evaluate the following functions :

2. $\dfrac{\varepsilon^x - \varepsilon^{-x}}{\log(1+x)}$, when $x = 0$. $\quad 2$.

3. $\dfrac{a^n - x^n}{\log a - \log x}$, $\quad x = a$. $\quad na^n$.

4. $\dfrac{x^3 - 5x^2 + 7x - 3}{x^3 - x^2 - 5x - 3}$, $\quad x = 3$. $\quad \dfrac{1}{4}$.

5. $\dfrac{x^4 - 8x^3 + 22x^2 - 24x + 9}{x^4 - 4x^3 - 2x^2 + 12x + 9}$, $\quad x = 3$. $\quad \dfrac{1}{4}$.

6. $\dfrac{x\varepsilon^{2x} - \varepsilon^{2x} - x + 1}{\varepsilon^{2x} - 1}$, $\quad x = 0$. $\quad -1$.

7. $\dfrac{\sin x - \cos x}{\sin 2x - \cos 2x - 1}$, when $x = \frac{1}{4}\pi$. $\quad \frac{1}{2}\sqrt{2}$.

8. $\dfrac{\log x}{\sqrt{(1 - x)}}$, $\quad x = 1$. $\quad 0$.

9. $\dfrac{a^x - b^x}{x}$, $\quad x = 0$. $\quad \log \dfrac{a}{b}$.

10. $\dfrac{\sqrt{(1 + x^2)}\,(1 - x)}{1 - x^n}$, (*See Art.* 98), $\quad x = 1$. $\quad \dfrac{\sqrt{2}}{n}$.

11. $\dfrac{a^2 - x^2}{x^2}\left(1 - \cos\dfrac{x}{a}\right)$, $\quad x = 0$. $\quad \dfrac{1}{2}$.

12. $\dfrac{\varepsilon^{mx} - \varepsilon^{ma}}{x - a}$, $\quad x = a$. $\quad m\varepsilon^{ma}$.

13. $\dfrac{a^{\sin x} - a}{\log \sin x}$, $\quad x = \frac{1}{2}\pi$. $\quad a \log a$.

14. $\dfrac{1 - \cos x}{x \log(1 + x)}$, $\quad x = 0$. $\quad \dfrac{1}{2}$.

15. $\dfrac{\sqrt{x} \tan x}{(\varepsilon^x - 1)^{\frac{3}{2}}}$, $\quad x = 0$. $\quad 1$.

Put in the form $\sqrt{\dfrac{x}{\varepsilon^x - 1}} \cdot \dfrac{\tan x}{x} \cdot \dfrac{x}{\varepsilon^x - 1}$. *See Art.* 99 *and Example* 1.

16. $\dfrac{\sqrt{x} - \sqrt{a} + \sqrt{(x - a)}}{\sqrt{(x^2 - a^2)}}$, $\quad x = a$. $\quad \dfrac{1}{\sqrt{(2a)}}$.

17. $\dfrac{x\sqrt{(3x - 2x^4)} - x^{\frac{6}{5}}}{1 - x^{\frac{2}{3}}}$, $\quad x = 1$. $\quad \dfrac{81}{20}$.

18. $\dfrac{(a^2 + ax + x^2)^{\frac{1}{2}} - (a^2 - ax + x^2)^{\frac{1}{2}}}{(a + x)^{\frac{1}{2}} - (a - x)^{\frac{1}{2}}}$, $\quad x = 0$. $\quad \sqrt{a}$.

Multiply both terms by the two complementary surds. See Art. 94.

19. $\dfrac{(a^2 - x^2)^{\frac{1}{2}} + (a - x)^{\frac{3}{2}}}{(a^3 - x^3)^{\frac{1}{2}} + (a - x)^{\frac{1}{2}}}$, when $x = a$. $\dfrac{\surd(2a)}{1 + a\surd 3}$.

Divide both terms by $(a - x)^{\frac{1}{2}}$.

20. $\dfrac{\sin x - x \cos x}{x - \sin x}$, $x = 0$. 2.

✓ 21. $\dfrac{\varepsilon^x - \varepsilon^{-x} - 2x}{x - \tan x}$, $x = 0$. -1. ✓

22. $\dfrac{(x - 2)\varepsilon^x + x + 2}{x(\varepsilon^x - 1)^2}$, $x = 0$. $\dfrac{1}{6}$.

23. $\dfrac{x^x - x}{1 - x + \log x}$, $x = 1$. -2.

24. $\dfrac{\tan x - \sin x}{x^3}$, $x = 0$. $\dfrac{1}{2}$.

Put in the form $\left.\dfrac{\sin x}{x}\right]_0 \cdot \left.\dfrac{\sec x - 1}{x^2}\right]_0$.

25. $\dfrac{(x - 1)^2 + \sin^2(x^2 - 1)^{\frac{1}{2}}}{(x + 1)\,(x - 1)^{\frac{3}{2}}}$, $x = 1$. $\surd 2$.

26. $\dfrac{1 - x + \log x}{1 - \surd(2x - x^2)}$, $x = 1$. -1.

27. $\dfrac{\sin x - \log(\varepsilon^x \cos x)}{x^2}$, $x = 0$. $\dfrac{1}{2}$.

28. $\dfrac{\frac{1}{4}\pi - \tan^{-1} x}{x^n - \varepsilon^{\sin(\log x)}}$, $x = 1$. $\dfrac{1}{2(1 - n)}$.

29. $\dfrac{\tan(a + x) - \tan(a - x)}{\tan^{-1}(a + x) - \tan^{-1}(a - x)}$, $x = 0$. $(1 + a^2)\sec^2 a$.

30. $\dfrac{x \sin x - \frac{1}{2}\pi}{\cos x}$, $x = \frac{1}{2}\pi$. -1.

31. $\dfrac{\varepsilon^x - \varepsilon^{\sin x}}{x - \sin x}$, when $x = 0$. 1.

32. $\dfrac{a^{\log x} - x}{\log x}$, $x = 1$. $\log a - 1$.

33. $\dfrac{x^{\frac{3}{2}} - 1 + (x-1)^{\frac{3}{2}}}{(x^2 - 1)^{\frac{3}{2}} - x + 1}$, $x = 1$. $-\dfrac{3}{2}$.

34. $\dfrac{\cos^{-1}(1-x)}{\sqrt{(2x - x^2)}}$, $x = 0$. 1.

35. $\dfrac{x^2 - a\sqrt{ax}}{\sqrt{(ax)} - a}$, $x = a$. $3a$.

36. $\dfrac{\tan nx - n\tan x}{n\sin x - \sin nx}$, $x = 0$. 2.

37. $\dfrac{\sqrt{2} - \cos x - \sin x}{\log \sin 2x}$, $x = \frac{1}{4}\pi$. $-\frac{1}{4}\sqrt{2}$.

38. $\dfrac{x + x^3 - (2n+1)x^{2n+1} + (2n-1)x^{2n+3}}{(1-x^2)^2}$, $x = 1$. n^2.

39. $\dfrac{m^n \sin nx - n^m \sin mx}{\tan nx - \tan mx}$, $x = 0$. 1.

40. $\dfrac{\tan nx - \tan mx}{\sin(n^2x - m^2x)}$, $x = 0$. $\dfrac{1}{m+n}$.

41. $\dfrac{m^n \sin nx - n^m \sin mx}{\tan nx - \tan mx}$, $m = n$.

$n^{n-1}(n\cos nx - \sin nx)\cos^2 nx$.

In solving this and the following example, x *and* n *may be regarded as constants, and* m *as a variable.*

42. $\dfrac{\tan nx - \tan mx}{\sin(n^2x - m^2x)}$, when $m = n$. $\dfrac{\sec^2 nx}{2n}$.

43. Prove that, if $f(0) = 0$, $\left.\dfrac{\varepsilon^x - \varepsilon^{f(x)}}{x - f(x)}\right]_0 = 1$.

The given function may be put in the form

$$\varepsilon^{f(x)}\frac{\varepsilon^{x-f(x)} - 1}{x - f(x)};$$

when x = 0, *the second factor is unity by Example* 1.

XIV.

The Form $\dfrac{\infty}{\infty}$.

101. Let $\dfrac{f(x)}{\phi(x)}$ denote a function which assumes the form $\dfrac{\infty}{\infty}$ when $x = a$, then we have

$$\frac{f(x)}{\phi(x)} = \frac{\dfrac{1}{\phi(x)}}{\dfrac{1}{f(x)}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

The second member of this equation takes the form $\dfrac{0}{0}$ when $x = a$; we therefore have, by equation (1) Art. 96,

$$\frac{f(a)}{\phi(a)} = \frac{\dfrac{1}{\phi(a)}}{\dfrac{1}{f(a)}} = \frac{-\dfrac{\phi'(a)}{[\phi(a)]^2}}{-\dfrac{f'(a)}{[f(a)]^2}} = \frac{\phi'(a)}{f'(a)}\left\{\frac{f(a)}{\phi(a)}\right\}^2; \quad . \quad . \qquad (2)$$

whence, if $\dfrac{f(a)}{\phi(a)}$ is neither zero nor infinity, we infer that

$$\frac{f(a)}{\phi(a)} = \frac{f'(a)}{\phi'(a)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This formula, it will be observed, is identical with that employed when the function takes the form $\frac{0}{0}$.

102. In the above demonstration, it is assumed that both $\frac{f(a)}{\phi(a)}$ and $\frac{f'(a)}{\phi'(a)}$ have definite values; and also that the required value is neither zero nor infinity. It will be shown in the next article that the latter assumption is not essential to the existence of equation (3); but when the *former* assumption does not hold true, this equation ceases to be applicable. Thus, in the case of the function

$$\frac{x - \sin x}{x + \cos x},*$$

which takes the form $\frac{\infty}{\infty}$, when $x = \infty$, we have

$$\frac{f'(x)}{\phi'(x)} = \frac{1 - \cos x}{1 - \sin x}.$$

An expression which has no definite value when x is infinite, since $\sin \infty$ and $\cos \infty$ have no definite values. The given function has, nevertheless, the definite value unity; for

$$\left.\frac{x - \sin x}{x + \cos x}\right]_{\infty} = \left.\frac{1 - \dfrac{\sin x}{x}}{1 + \dfrac{\cos x}{x}}\right]_{\infty} = 1.$$

103. When the value of $\frac{f(a)}{\phi(a)}$ is either zero or infinity, equa-

* *Calcul Différentiel*, par J. Bertrand, Paris, 1864, p. 476.

tion (2), Art. 101, will be satisfied independently of the existence of equation (3); we are not justified therefore, when this is the case, in deriving the latter from the former. The following demonstration shows, however, that equation (3) holds in these cases also.

First, when the value of $\frac{f(a)}{\phi(a)}$ is zero, by adding a finite quantity n to the given function, we have

$$\frac{f(a)}{\phi(a)} + n = \frac{f(a) + n\phi(a)}{\phi(a)},$$

a function which is by hypothesis finite. To this function there fore the demonstration given in Art. 101 applies; hence

$$\frac{f(a)}{\phi(a)} + n = \frac{f'(a) + n\phi'(a)}{\phi'(a)} = n + \frac{f'(a)}{\phi'(a)};$$

therefore
$$\frac{f(a)}{\phi(a)} = \frac{f'(a)}{\phi'(a)},$$
as before.

Again, if the value of $\frac{f(a)}{\phi(a)}$ is infinite, that of $\frac{\phi(a)}{f(a)}$ is zero, and, by the last result,

$$\frac{\phi(a)}{f(a)} = \frac{\phi'(a)}{f'(a)};$$

hence, in this case, likewise

$$\frac{f(a)}{\phi(a)} = \frac{f'(a)}{\phi'(a)}.$$

Derivatives of Functions which assume an Infinite Value.

104. *When* f(x) *becomes infinite, for a finite value* a *of the independent variable,* f'(a) *is likewise infinite.* For, let b denote a

value of x so taken that $f(x)$ shall be finite for $x = b$ and for all values of x between b and a: then, as x varies from b to a, the rate of $f(x)$ must assume an infinite value, otherwise $f(x)$ would remain finite. The value of x for which the rate is infinite must be a or some value of x between b and a; that is, some value of x nearer to a than b is. Now, since b may be taken as near as we please to a, the value of x for which the rate is infinite cannot differ from a. The expression for this rate is $f'(x)\dfrac{dx}{dt}$, in which $\dfrac{dx}{dt}$ may be assumed finite, therefore $f'(x)$ must be infinite when $x = a$; in other words, f′(a) *is infinite when* f(a) *is infinite.*

105. It follows from the theorem proved in the preceding article that when a is finite the function obtained by the application of formula (3), Art. 101, takes the same form, $\dfrac{\infty}{\infty}$, as that assumed by the original function. Hence, except when the given value of x is infinite, the application of some other process, either to the original function or to one of the substituted functions, is always requisite. Thus in the example,

$$\left.\frac{\log(\sin 2x)}{\log \sin x}\right]_0 = \frac{\infty}{\infty};$$

by using the above formula we obtain

$$\left.\frac{\log \sin 2x}{\log \sin x}\right]_0 = \left.\frac{2 \cot 2x}{\cot x}\right]_0,$$

which takes the form $\dfrac{\infty}{\infty}$; but the last expression is equivalent to $\left.2\dfrac{\sin x \cos 2x}{\sin 2x \cos x}\right]_0$, and is therefore easily shown to have the value unity.

Algebraic Fractions, x *being Infinite.*

106. A fraction of the following general form;

$$\frac{ax^\alpha + bx^\beta + cx^\gamma + \cdots}{a'x^{\alpha'} + b'x^{\beta'} + c'x^{\gamma'} + \cdots},$$

in which α and α' denote the highest exponents in the numerator and in the denominator respectively, takes the illusory form $\frac{\infty}{\infty}$ when x becomes infinite, provided α and α' are positive. To determine the value of this fraction, we put it in the form

$$\frac{x^\alpha(a + bx^{\beta-\alpha} + cx^{\gamma-\alpha} + \cdots)}{x^{\alpha'}(a' + b'x^{\beta'-\alpha'} + c'x^{\gamma'-\alpha'} + \cdots)},$$

in which the exponents $\beta - \alpha$, $\gamma - \alpha$, $\beta' - \alpha'$, etc., are all negative. If now we make x infinite, all the terms within the marks of parenthesis, except the first, will vanish; accordingly, the value of the fraction for this value of x is identical with that of

$$\frac{a}{a'}\, x^{\alpha-\alpha'}.$$

When $\alpha > \alpha'$ this value is infinite, when $\alpha < \alpha'$ it is zero, and when $\alpha = \alpha'$ it reduces to the finite quantity $\frac{a}{a'}$. In either case, *the value equals the ratio of the term of the highest degree in the numerator to that in the denominator.*

For example,

$$\left.\frac{4x^6 + 5x^4 + 3x + 10}{8x^6 - 32}\right]_\infty = \frac{1}{2}.$$

The Form 0 . ∞.

107. A function which takes this form may, by introducing the reciprocal of one of the factors, be so transformed as to take

either of the forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$, as may be found most convenient. For example, let us take the function

$$x^{-n}\varepsilon^x,$$

which assumes the above form when $x = \infty$, n being positive. In this case it is necessary to reduce to the form $\frac{\infty}{\infty}$. Thus—

$$x^{-n}\varepsilon^x = \frac{\varepsilon^x}{x^n}\Big]_\infty = \frac{\varepsilon^x}{nx^{n-1}}\Big]_\infty = \frac{\varepsilon^x}{n(n-1)x^{n-2}}\Big]_\infty, \text{ etc.}$$

By continuing this process, we finally obtain a fraction whose denominator is finite while its numerator is still infinite. Hence we have, for all finite values of n,

$$x^{-n}\varepsilon^x\Big]_\infty = \infty.$$

The Form $\infty - \infty$.

108. A function which assumes this form may be so transformed as to take the form $\frac{0}{0}$. Let the given function be

$$\left[\frac{1}{x(1+x)} - \frac{\log(1+x)}{x^2}\right]_0,$$

which takes the form $\infty - \infty$, since the second term is easily shown to be infinite. But

$$\left[\frac{1}{x(1+x)} - \frac{\log(1+x)}{x^2}\right]_0 = \frac{x-(1+x)\log(1+x)}{x^2(1+x)}\Big]_0$$

$$= \frac{x-(1+x)\log(1+x)}{x^2}\Big]_0$$

$$= \frac{1-\log(1+x)-1}{2x}\Big]_0 = -\frac{1}{2}.$$

Examples XIV.

Evaluate the following functions:

1. $\dfrac{\sec x}{\sec 3x}$,	when $x = \frac{1}{2}\pi$.	-3.
2. $\dfrac{a^x}{\operatorname{cosec}(ma^{-x})}$,	$x = \infty$.	m.
3. $\dfrac{\log x}{x^n}$ $(n > 0)$,	$x = \infty$.	0.
4. $\dfrac{\tan x}{\log(x - \frac{1}{2}\pi)}$,	$x = \frac{1}{2}\pi$.	∞.
5. $\dfrac{\sec(\frac{1}{2}\pi x)}{\log(1 - x)}$,	$x = 1$.	∞.
6. $\dfrac{\log\cos(\frac{1}{2}\pi x)}{\log(1 - x)}$,	$x = 1$.	1.
7. $\dfrac{\tan x}{\tan 3x}$,	$x = \frac{1}{2}\pi$.	3.
8. $\dfrac{\log(1 + x)}{x}$,	$x = \infty$.	0.
9. $\left(a^{\frac{1}{x}} - 1\right)x$,	$x = \infty$.	$\log a$.
10. $\dfrac{x^2 - a^2}{a^2}\tan\dfrac{\pi x}{2a}$,	$x = a$.	$-\dfrac{4}{\pi}$.
11. $x^m(\log x)^n$ (m *and* n *being positive*),	$x = 0$.	0.
12. $e^x \sin\dfrac{1}{x}$,	$x = \infty$.	∞.

13. $\varepsilon^{-\frac{1}{x}}(1 - \log x)$,	$x = 0$.	0.
14. $\sec\dfrac{\pi x}{2} \cdot \log\dfrac{1}{x}$,	$x = 1$.	$\dfrac{2}{\pi}$.
15. $\dfrac{\log\tan nx}{\log\tan x}$,	$x = 0$.	1.
16. $\dfrac{\log\cot\frac{x}{2}}{\cot x + \log x}$,	$x = 0$.	0.
17. $\sec x\,(x\sin x - \frac{1}{2}\pi)$,	$x = \frac{1}{2}\pi$.	-1.
18. $\log\left(2 - \dfrac{x}{a}\right)\tan\dfrac{\pi x}{2a}$,	$x = a$.	$\dfrac{2}{\pi}$.
19. $(1 - x)\tan(\frac{1}{2}\pi x)$,	$x = 1$.	$\dfrac{2}{\pi}$.
20. $\log(x - a)\tan(x - a)$,	$x = a$.	0.
21. $(a^2 - x^2)^{\frac{1}{2}}\cot\left\{\dfrac{\pi}{2}\left(\dfrac{a - x}{a + x}\right)^{\frac{1}{2}}\right\}$,	$x = a$.	

Denoting the arc by θ, *and multiplying by* $\dfrac{\tan\theta}{\theta}$ (*whose value, when* $x = a$, *is unity*) *we obtain* $\left.\dfrac{2}{\pi}(a + x)\right] = \dfrac{4a}{\pi}$.

22. $x\varepsilon^{\left(\frac{1}{x} - x\right)}$,	$x = 0$.	∞.
23. $\dfrac{\sec^n x}{\varepsilon^{\tan x}}$,	$x = \frac{1}{2}\pi$.	0.
24. $x - x^2\log\left(1 + \dfrac{1}{x}\right)$,	$x = \infty$.	$\dfrac{1}{2}$.

Put $x = \dfrac{1}{z}$.

25. $\dfrac{x(\varepsilon^{2x}+\varepsilon^{x})}{(\varepsilon^{x}-1)^{3}}-\dfrac{2\varepsilon^{x}}{(\varepsilon^{x}-1)^{2}}$,	$x=0$.	$\dfrac{1}{6}$.
26. $\dfrac{\operatorname{cosec} x}{x}-\dfrac{\sin^{-1} x}{x^{2}\sin x}$,	$x=0$.	$-\dfrac{1}{6}$.
27. $\dfrac{2}{x}-\cot \tfrac{1}{2}x$,	$x=0$.	0.
28. $x\tan x-\tfrac{1}{2}\pi\sec x$,	$x=\tfrac{1}{2}\pi$.	-1.
29. $\dfrac{x}{x-1}-\dfrac{1}{\log x}$,	$x=1$.	$\dfrac{1}{2}$.
30. $\dfrac{x}{\sin^{3} x}-\cot^{2} x$,	$x=0$.	$\dfrac{7}{6}$.
31. $\dfrac{1}{2x^{2}}-\dfrac{\pi}{2x\tan \pi x}$,	$x=0$.	$\tfrac{1}{6}\pi^{2}$.
32. $\dfrac{\pi}{4x}-\dfrac{\pi}{2x\,(\varepsilon^{\pi x}+1)}$,	$x=0$.	$\tfrac{1}{8}\pi^{2}$.
33. $\dfrac{\pi x-1}{2x^{2}}+\dfrac{\pi}{x(\varepsilon^{2\pi x}-1)}$,	$x=0$.	$\tfrac{1}{6}\pi^{2}$.

34. Prove that, when $f(a)=1$ and $\phi(a)=1$,

$$\frac{\log f(a)}{\log \phi(a)}=\frac{f'(a)}{\phi'(a)}.$$

35. Prove that, when $f(a)$ and $\phi(a)=0$,

$$\frac{\log f(a)}{\log \phi(a)}=1;$$

provided that $\dfrac{f'(a)}{\phi'(a)}$ is neither infinite nor zero.

XV.

Functions whose Logarithms take the Form $\infty \cdot 0$.

109. In the case of a function of the form u^v, we have

$$\log u^v = v \log u.$$

The expression $v \log u$ takes the illusory form $0 \cdot \infty$ in two cases: first, when $v = 0$ and $\log u = \infty$; and secondly, when $v = \infty$ and $\log u = 0$.

Log u is infinite when $u = 0$, and also when $u = \infty$; therefore the first case will arise when the original function takes one of the forms ∞^0 or 0^0.

Log $u = 0$ when $u = 1$, therefore the second case will arise when the original function takes the form 1^∞.

Hence functions which take either of the three illusory forms

$$\infty^0, \qquad 0^0, \qquad \text{or} \qquad 1^\infty,$$

may be evaluated by first evaluating their logarithms, which take the form $0 \cdot \infty$.

It is to be noticed however that 0^∞ and ∞^∞ are not illusory forms, since their logarithms take the form $\infty(\mp \infty)$.

The Form 1^∞.

110. As an illustration of this form, we take the function $\left(1 + \dfrac{a}{x}\right)^x$, which assumes the form 1^∞ when $x = \infty$. Denoting this function by u, we have

$$\log u = x \log\left(1 + \frac{a}{x}\right) = \frac{\log\left(1 + \dfrac{a}{x}\right)}{\dfrac{1}{x}},$$

the last expression assuming the form $\dfrac{0}{0}$ when $x = \infty$.

In evaluating this logarithm, it is convenient to substitute z for $\frac{1}{x}$; then

$$\log u_\infty = \frac{\log(1+az)}{z}\Big]_0,$$

since, when $x = \infty$, $z = 0$. Taking derivatives, we have

$$\log u_\infty = \frac{\log(1+az)}{z}\Big]_0 = \frac{a}{1+az}\Big]_0 = a.$$

Hence $$u_\infty = \left(1 + \frac{a}{x}\right)^x\Big]_\infty = \epsilon^a.$$

111. If $a = 1$, we have

$$\left(1 + \frac{1}{x}\right)^x\Big]_\infty = \epsilon;$$

that is, as x increases indefinitely, the *limiting value* of the function $\left(1 + \frac{1}{x}\right)^x$ is ϵ. The Napierian base is often defined as the limiting value of this function, or, what is the same thing, by formula

$$\epsilon = (1 + x)^{\frac{1}{x}}\Big]_0.$$

The Form 0^0.

112. The function $x^x]_0$, by the aid of which many functions of similar form may be evaluated, will serve as an illustration of the form 0^0.

Let $$u = x^x;$$

then $$\log u = x \log x,$$

and $$\log u\Big]_0 = \frac{\log x}{x^{-1}}\Big]_0 = -\frac{x^{-1}}{x^{-2}}\Big]_0 = 0;$$

therefore $$x^x\Big]_0 = \epsilon^0 = 1.$$

The value of a function which takes the form 0^0 is usually found, as in the above example, to be unity. This is not, however, *universally* true, as the function

$$x^{\frac{a+x}{\log x}}$$

(one of those earliest adduced for this purpose*) will show.

This function takes the form 0^0, when $x = 0$; but since its logarithm reduces to $a + x$, its value when $x = 0$ is ϵ^a.

Functions having two limiting values for a single value of x.

113. An exponential function may have two limiting values corresponding to that value of x for which the exponent becomes infinite; when this is the case the function is said to be discontinuous.

Thus the function $\epsilon^{\frac{1}{x}}$ increases without limit when x is positive and approaches zero; but when x is negative, denoting its numerical value by x', we have

$$\epsilon^{\frac{1}{x}} = \epsilon^{-\frac{1}{x'}},$$

which becomes zero when x' vanishes. These results may be expressed thus—

$$\epsilon^{\frac{1}{x}}\Big]_0 = \infty, \text{ but } \epsilon^{-\frac{1}{x'}}\Big]_0 = 0,$$

it being understood that x in the first expression is positive

* See *Crelle's Journal*, vol. xii, p. 293.

until it reaches the value zero, and that in the second expression x' is positive until it reaches zero. The curve $y = \varepsilon^x$ is traced in Art. 247.

114. The same peculiarity belongs also to the function

$$u = x^n \varepsilon^{\frac{1}{x}},$$

n having any finite value. For, supposing x to be positive, and denoting by u_0 the value of the function when x vanishes, u_0 is evidently infinite when n is negative, but takes the form $0 \cdot \infty$ when n is positive. To evaluate u_0 in this case we put $x = \frac{1}{z}$; whence

$$u_0 = x^n \varepsilon^{\frac{1}{x}}\Big]_0 = \frac{\varepsilon^z}{z^n}\Big]_\infty = \frac{\infty}{\infty}.$$

This function has been shown, in Art. 107, to have an infinite value; therefore, we have for all values of n (x being positive)

$$u_0 = \infty.$$

Now, when x is negative, let $x = -x'$, then

$$u = (-x')^n \varepsilon^{-\frac{1}{x'}}.$$

Denoting the value which this function assumes when x' is positive and vanishes by u'_0, we have

$$u'_0 = \frac{1}{(-x')^{-n} \varepsilon^{\frac{1}{x'}}}.$$

The denominator of this fraction is infinite for all values of n by the demonstration given above; hence we have, when x is negative, for all values of n

$$u'_0 = 0.$$

Examples XV.

1. $(\cos x)^{\cot^2 x}$, when $x = 0$. $\varepsilon^{-\frac{1}{2}}$.

2. $\left(\dfrac{\tan x}{x}\right)^{\frac{1}{x^2}}$, $x = 0$. $\sqrt[3]{\varepsilon}$.

3. $(\cos \alpha x)^{\operatorname{cosec}^2 \beta x}$, $x = 0$. $\varepsilon^{-\frac{a^2}{2\beta^2}}$.

4. $\left(\dfrac{1}{x}\right)^{\tan x}$, $x = 0$. 1.

5. $(\tan x)^{\tan 2x}$, $x = -\frac{1}{4}\pi$. 1.

6. $\left(\dfrac{1}{x^n}\right)^{x^m} (m > 0)$, $x = 0$. 1.

7. $(1 - x)^{\frac{1}{x}}$, $x = 0$. $\dfrac{1}{\varepsilon}$.

8. $(\sin x)^{\sec^2 x}$, $x = \frac{1}{2}\pi$. $\varepsilon^{-\frac{1}{2}}$.

9. $(\cot x)^{\sin x}$, $x = 0$.

Solution: $(\cot x)^{\sin x}\Big]_0 = \dfrac{(\cos x)^{\sin x}\big]_0}{(\sin x)^{\sin x}\big]_0} = 1$. (*See Art.* 112.)

10. $(\sin x)^{\tan x}$, $x = 0$. 1.

11. $(\sin x)^{\tan x}$, $x = \dfrac{\pi}{2}$. 1.

12. $x^{\frac{a}{\log \sin x}}$, $x = 0$. ε^{a}.

13. $(\sin x)^{\frac{a^2 - x^2}{\log \tan x}}$, $x = 0$. ε^{a^2}.

14. $x^{x^a} (a > 0)$, $x = 0$. 1.

15. $(x^2)^{\frac{(a+x)^2}{\log (x + \log \cos x)}}$, $x = 0$. ε^{2a^2}.

16. $x^{\frac{1}{1-x}}$, when $x = 1$. $\quad \dfrac{1}{e}$.

17. $x^{x^x - 1}$, $\quad x = 0$. $\quad 1$.

18. $(\cos mx)^{\frac{n}{x^2}}$, $\quad x = 0$. $\quad e^{-\frac{1}{2}nm^2}$.

19. $\left(\dfrac{\log x}{x}\right)^{\frac{1}{x}}$, $\quad x = \infty$. $\quad 1$.

20. $(1 \pm x)^{\frac{1}{x}}$, $\quad x = \infty$. $\quad 1$.

21. $x^m (\sin x)^{\tan x}\left(\dfrac{\pi - 2x}{2 \sin 2x}\right)^3$, $\quad x = \dfrac{\pi}{2}$. $\quad \dfrac{\pi^m}{2^{m+3}}$.

22. $\dfrac{(m^x - 1)(a \sin x - \sin ax)^n}{x^n \sin x (\cos x - \cos ax)^n}$, $\quad x = 0$. $\quad \left(\dfrac{a}{3}\right)^n \log m$.

23. $\dfrac{(1 - x)^{\frac{1}{x}}}{x^2 (x - 1)}\Big[(1 - x)\log(1 - x) + x\Big]$, $\quad x = 0$. $\quad -\dfrac{1}{2e}$.

24. $\dfrac{(1 - x)^{\frac{1}{x}}}{x^2 (x - 1)}\Big[(1 - x)\log(1 - x) + x\Big]$, $\quad x = 1$. $\quad -1$.

25. $\dfrac{(1 + x)^{\frac{1}{x}} - e}{x}$, $\quad x = 0$.

Solution:—

Denoting $(1 + x)^{\frac{1}{x}}$ by y, and observing that, by Art. 111, when $x = 0$, $y = e$; it is evident that the function takes the form $\frac{0}{0}$; hence, on applying the usual process, its value is $\left.\dfrac{dy}{dx}\right]_0$. Differentiating, we have

$$\frac{dy}{dx} = y\left[\frac{1}{x(1 + x)} - \frac{\log(1 + x)}{x^2}\right] = yz,$$

in which z denotes the function evaluated in Art. 108, hence $z_0 = -\frac{1}{2}$. Therefore,

$$\left.\frac{(1+x)^{\frac{1}{x}} - \varepsilon}{x}\right]_0 = \left.\frac{dy}{dx}\right]_0 = -\frac{\varepsilon}{2}.$$

26. $\dfrac{x(1+x)^{\frac{1}{x}} - \varepsilon \sin x)}{x^2}$, when $x = 0$.

Solution :—

Using the notation and results of the preceding example, we have

$$\left.\frac{xy - \varepsilon \sin x}{x^2}\right]_0 = \left.\frac{y + x\dfrac{dy}{dx} - \varepsilon \cos x}{2x}\right]_0$$

$$= \left.\frac{y - \varepsilon \cos x}{2x}\right]_0 + \frac{1}{2}\left.\frac{dy}{dx}\right]_0$$

$$= \left.\frac{\dfrac{dy}{dx} - \varepsilon \sin x}{2}\right]_0 - \frac{\varepsilon}{4} = -\frac{\varepsilon}{2}$$

27. z being defined as in Example 25, prove that

$$\left.\frac{dz}{dx}\right]_0 = \frac{2}{3}.$$

28. $\dfrac{(1+x)^{\frac{1}{x}} - \varepsilon + \dfrac{\varepsilon x}{2}}{x^2}$, when $x = 0$. $\dfrac{11\varepsilon}{24}$.

Use the notation and results of Examples 25 *and* 27.

29. $\dfrac{1}{x^2}\left[(1+x)^{\frac{1}{x}} - \dfrac{\varepsilon}{x}\log(1+x)\right]$, when $x = 0$. $\frac{1}{8}\varepsilon$.

Put in the form $\dfrac{y - \varepsilon x^{-1}\log(1+x)}{x^2}$. (*See Example* 25.)

30. If $y = \dfrac{1}{1 + \epsilon^{\frac{1}{x}}}$, give the value of y, and also that of $\dfrac{dy}{dx}$, when x approaches zero from the positive and from the negative side. (*See Art.* 113.)

Result, (positive side) : $y = 0$; $\dfrac{dy}{dx} = 0$;

(negative side) : $y = 1$; $\dfrac{dy}{dx} = 0$.

XVI.

Indeterminate Forms of Functions of Two Variables.

115. When a function of two independent variables takes an illusory form, its value is in general indeterminate. For example, the function

$$\frac{x^2 - 3xy + x}{y^2 - xy + 1}$$

takes the form $\frac{0}{0}$, when $x = 2$ and $y = 1$. Employing the usual method, we obtain

$$\left.\frac{x^2 - 3xy + x}{y^2 - xy + 1}\right]_{2,1} = \left.\frac{2x - 3y + 1 - 3x\dfrac{dy}{dx}}{(2y - x)\dfrac{dy}{dx} - y}\right]_{2,1} = 6\left.\frac{dy}{dx}\right]_{2,1} - 2.$$

The value of this expression depends upon that of $\dfrac{dy}{dx}$, and hence is *really indeterminate* when x and y are independent; but, when x and y are connected by an equation, $\dfrac{dy}{dx}$ has a definite value

for the given values of x and y, and consequently the given function has also a definite value.

116. In certain cases, the value of a function of two variables which takes an illusory form is independent of that of $\frac{dy}{dx}$; thus,

$$\frac{\log x + \log y}{x + y - 2}$$

takes an indeterminate form, when $x = 1$ and $y = 1$. Evaluating this expression we obtain

$$\left.\frac{\frac{1}{x} + \frac{1}{y}\frac{dy}{dx}}{1 + \frac{dy}{dx}}\right]_{1,1} = \frac{1 + \left.\frac{dy}{dx}\right]_{1,1}}{1 + \left.\frac{dy}{dx}\right]_{1,1}} = 1,$$

which is independent of $\left.\frac{dy}{dx}\right]_{1,1}$.

The Evaluation of Derivatives of Implicit Functions.

117. When y is an implicit function of x, its derivative is an expression involving both x and y; hence, if it takes an indeterminate form for given values of x and y, the expression obtained by the usual process of evaluation will involve the required derivative. Thus, given

$$y^3 + 3ay^2 - 2axy - ax^2 = 0, \quad . \quad . \quad . \quad . \quad (1)$$

to find the value of $\frac{dy}{dx}$, when $x = 0$ and $y = 0$, these being values which satisfy equation (1). In this case

$$\frac{dy}{dx} = \frac{2ay + 2ax}{3y^2 + 6ay - 2ax},$$

which takes the form $\frac{0}{0}$ for $x = 0$ and $y = 0$. Evaluating, we obtain

$$\left.\frac{dy}{dx}\right]_{0,0} = \frac{2a\left.\frac{dy}{dx}\right]_{0,0} + 2a}{6a\left.\frac{dy}{dx}\right]_{0,0} - 2a} = \frac{\left.\frac{dy}{dx}\right]_{0,0} + 1}{3\left.\frac{dy}{dx}\right]_{0,0} - 1},$$

an equation involving the required derivative in both members. Clearing this equation of fractions, we obtain the quadratic

$$3\left.\frac{dy}{dx}\right]_{0,0}^2 - 2\left.\frac{dy}{dx}\right]_{0,0} = 1,$$

whence we have

$$\left.\frac{dy}{dx}\right]_{0,0} = 1, \text{ and } \left.\frac{dy}{dx}\right]_{0,0} = -\tfrac{1}{3}.$$

This result shows that, regarding (1) as the equation of a curve, there are two values of $\frac{dy}{dx}$, or $\tan\phi$, at the origin, indicating that the curve passes *twice* through the origin.

118. When $x = 0$ and $y = 0$ are simultaneous values of x and y, the ratio $\frac{y}{x}$ takes the form $\frac{0}{0}$, and by the formula for evaluation we have

$$\left.\frac{y}{x}\right]_{0,0} = \left.\frac{dy}{dx}\right]_{0,0}.$$

When the value of $\left.\frac{dy}{dx}\right]_{0,0}$ is required, we may therefore employ the expression $\left.\frac{y}{x}\right]_{0,0}$ which often admits of evaluation by an

algebraic process. Thus dividing equation (1) of the preceding article by x^2 we have

$$y\frac{y^2}{x^2} + 3a\frac{y^2}{x^2} - 2a\frac{y}{x} - a = 0,$$

in which if we put $y = 0$, and assume the value of $\left.\frac{y}{x}\right]_0$ to be finite, we obtain

$$3\left.\frac{y}{x}\right]_0^2 - 2\left.\frac{y}{x}\right]_0 - 1 = 0.$$

This quadratic is the same as the one employed in the preceding article to determine $\left.\frac{dy}{dx}\right]_0$.

It is obvious that in this process all the terms in the given equation, except those lowest in degree, disappear when we put $x = 0$ and $y = 0$; hence the same result may be obtained by putting the terms of lowest degree equal to zero. When $\left.\frac{y}{x}\right]_0$ is infinite the same reasoning is applicable, since we may in this case employ the reciprocal $\left.\frac{x}{y}\right]_0$.

In the given example we find two values of $\left.\frac{y}{x}\right]_0$ because the terms employed are of the second degree.

Examples XVI.

1. The variables x and y being connected by the equation

$$2(1 - x + y) - 4y^2 + 3x^4 = 0,$$

show that $x = 0$ and $y = 1$ are simultaneous values, and find the corresponding value of

$$\frac{y^3 + x^3 - 8x^2 + x - 1}{xy^2 - 4x^2}. \qquad \textit{Result } 0.$$

2. The relation between x and y being expressed by the equation

$$3x^4 + 4y^3 - 24(x + y) + 37 = 0,$$

show that $x = 1$ and $y = 2$ are simultaneous values, and find the corresponding value of

$$\frac{y^4 - 16x}{y^2 - x^2 - 3}. \qquad 5\tfrac{9}{19}.$$

In this example, on substituting the numerical value of $\frac{dy}{dx}$, *the function again takes the indeterminate form; it is therefore necessary to substitute the value of* $\frac{dy}{dx}$ *in terms of* x *and* y, *and to repeat the process.*

3. The relation between x and y being expressed by the equation

$$y^2 + x^2 - 2\epsilon x - \epsilon^2 = 0,$$

show that $x = 0$ and $y = \epsilon$ are simultaneous values, and find the corresponding value of

$$\frac{\epsilon^x - \log y}{y^x - \log(\epsilon - x)}. \qquad \frac{\epsilon - 1}{\epsilon + 1}.$$

4. Given $y^x = x^y$; find the values of $\frac{dy}{dx}$ when $x = \epsilon$ and $y = \epsilon$. 1.

5. Given $x^3 - 3axy + y^3 + a^3 = 0$;

show that $x = a$ and $y = a$ are simultaneous values, and find the corresponding values of $\frac{dy}{dx}$. $\frac{1}{2}[1 \pm \sqrt{(-3)}]$.

6. Given $x^3 - 3xy + y^3 = 0$,

find the values of $\frac{dy}{dx}$ when $x = 0$ and $y = 0$. 0 and ∞.

7. Given $$y^4 + 3a^2y^2 - 4a^2xy - a^2x^2 = 0;$$
find the values of $\dfrac{dy}{dx}$ when $x = 0$ and $y = 0$.

$\frac{1}{3}(2 \pm \sqrt{7})$.

8. Given $$x^3 - axy + a^2x - ay^2 + 2a^2y - a^3 = 0;$$
show that $x = 0$ and $y = a$ are simultaneous values, and find the corresponding values of $\dfrac{dy}{dx}$.

0, and -1.

9. Given $$y^4 - 96a^2y^2 + 100a^2x^2 - x^4 = 0;$$
find the values of $\dfrac{dy}{dx}$ corresponding to $x = 0$ and $y = 0$.

$\pm\frac{5}{12}\sqrt{6}$.

10. Given $$y^4 - x^4 - 4ay^3 + 5a^2y^2 + 2a^2x^2 - 2a^3y = 0;$$
show that $x = 0$ and $y = a$ are simultaneous values, and find the corresponding values of $\dfrac{dy}{dx}$.

$\pm\sqrt{2}$.

11. Given $$x^4 - 2ay^3 - 3a^2y^2 - 2a^2x^2 + a^4 = 0;$$
find the values of $\dfrac{dy}{dx}$ when $y = 0$, also when $x = 0$.

$$\left.\frac{dy}{dx}\right]_{\pm a,0} = \pm\tfrac{2}{3}\sqrt{3}, \quad \left.\frac{dy}{dx}\right]_{0,-a} = \pm\tfrac{1}{3}\sqrt{6}, \quad \text{and} \quad \left.\frac{dy}{dx}\right]_{0,\frac{1}{2}a} = 0.$$

12. Given $$y^3 - a(x + a)(x + y) = 0;$$
find, by the method of Art. 118, the value of $\left.\dfrac{dy}{dx}\right]_{0,0}$.

$$\left.\frac{dy}{dx}\right]_{0,0} = -1.$$

13. Given $$x^4 + ax^2y - ay^3 = 0;$$
find the values of $\dfrac{dy}{dx}$ when $x = 0$.

$$\left.\frac{dy}{dx}\right]_{0,0} = 0 \quad \text{or} \quad \pm 1.$$

14. If $$y = x(x - 1)\log(x \pm \sqrt{x}),$$

find, by the method of Art. 118, the value of $\frac{dy}{dx}$ when $x = 0$; also its values when $x = 1$, by substituting x' for $x - 1$.

$$\left.\frac{dy}{dx}\right]_{0,0} = \infty, \qquad \left.\frac{dy}{dx}\right]_{1,0} = -\infty \text{ and } \log 2.$$

15. Prove by putting $y = xf(x)$, and comparing the values of $\left.\frac{dy}{dx}\right]_0$ as found by differentiation and by the method of Art. 118, that if $f(0)$ is finite $xf'(x) = 0$, when $x = 0$, even though $f'(0)$ is infinite.

CHAPTER VI.

MAXIMA AND MINIMA OF FUNCTIONS OF A SINGLE VARIABLE.

XVII.

Conditions Indicating the Existence of Maxima and Minima.

119. IF, while the independent variable increases continuously, a function dependent on it increases up to a certain value, and then decreases, this value of the function is said to be a ***maximum*** value. In other words, a function $f(x)$ has a maximum value corresponding to $x = a$, if, when x increases through the value a, the function changes from an increasing to a decreasing function.

Since $f'(x)$ is positive, when $f(x)$ is an increasing function, and negative when it is a decreasing function; it is obvious that if $f(a)$ is a maximum value of $f(x)$, $f'(x)$ must ***change sign***, from $+$ to $-$, as x increases through the value a.

On the other hand, a function is said to have a ***minimum*** value for $x = a$, if it is a decreasing function before x reaches this value and an increasing one afterward. In this case, $f'(x)$ changes sign from $-$ to $+$.

120. The derivative $f'(x)$ can only change sign on passing through zero or infinity. Hence a value of x, for which $f(x)$ is a maximum or a minimum, must satisfy one of the two following equations:

$$f'(x) = 0 \qquad \text{and} \qquad f'(x) = \infty.$$

The required values of x will therefore be found among the roots of these equations.

The case which usually presents itself, and which will therefore be considered first, is that in which the required value of x is a root of the equation $f'(x) = 0$.

121. As an illustration, let it be required *to divide a number into two such parts that the square of one part multiplied by the cube of the other shall give the greatest possible product.*

Denote the given number by a, and the part to be squared by x; then we have

$$f(x) = x^2 (a - x)^3.$$

It is evident that a maximum value of this function exists; for when $x = 0$ its value is zero, and when $x = a$ its value is again zero, while for intermediate values of x it is positive; hence the function must change from an increasing to a decreasing function at least once, while x passes from the value zero to the value a.

Taking the derivative of this function, the equation

$$f'(x) = 0$$

is in this case $\quad 2x(a - x)^3 - 3x^2 (a - x)^2 = 0,$

or

$$x(a - x)^2 (2a - 5x) = 0.$$

0 and a are roots of this equation; but, as we are in search of a value of the function corresponding to an intermediate value of x, we put

$$2a - 5x = 0,$$

and obtain

$$x = \tfrac{2}{5}a.$$

The corresponding value of the function is $\frac{108}{3125}a^5$, the maximum value sought.

Maxima and Minima of Geometrical Magnitudes.

122. When the maximum or minimum value of a geometrical magnitude limited by certain conditions is required, it is necessary to obtain an expression for the magnitude in terms of a single unknown quantity, such that the determination of the value of this quantity will constitute the solution of the problem. For example: *let it be required to determine the cone of greatest convex surface among those which can be inscribed in a sphere whose radius is* a.

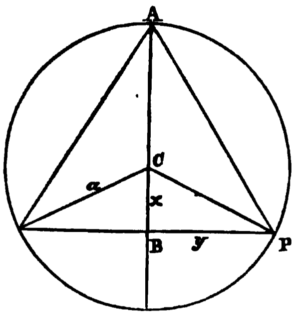


FIG. 10.

Any point A of the surface of the sphere being taken as the apex of the cone, let the diagram represent a great circle of the sphere passing through the fixed point A.

If we refer the position of the point P to rectangular coordinates, and take C as the origin, the required cone will evidently be determined when x is determined. We have now to express the convex surface S in terms of x.

The expression for the convex surface of a cone gives

$$S = \pi y \sqrt{[y^2 + (a + x)^2]}, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which the unknown quantities x and y are connected by the equation of the circle

$$x^2 + y^2 = a^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Substituting the value of y, we have

$$S = \pi \sqrt{(a^2 - x^2)} \sqrt{(2a^2 + 2ax)},$$

reducing, $$S = \pi \sqrt{(2a)} (a + x) \sqrt{(a - x)}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Since the factor $\pi\sqrt{(2a)}$ is constant, we are evidently required to find the value of x for which the function

$$f(x) = (a + x)\sqrt{(a - x)}$$

is a maximum. The equation $f'(x) = 0$ is, in this case,

$$\sqrt{(a - x)} - \frac{a + x}{2\sqrt{(a - x)}} = 0;$$

whence $$x = \tfrac{1}{3}a.$$

The altitude of the required cone is therefore $\frac{4}{3}a$. Substituting this value of x in equation (3), we have

$$S = \tfrac{8}{9}\sqrt{3}\cdot\pi a^2,$$

the maximum value required.

123. As a further illustration, let it be required *to determine the greatest cylinder that can be inscribed in a given segment of a paraboloid of revolution.*

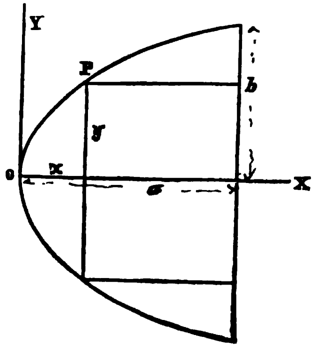


FIG 11.

Let a denote the altitude, and b the radius of the base of the segment. The equation of the generating parabola is of the form

$$y^2 = 4cx.$$

Since (a, b) is a point of the curve, we have the condition

$$b^2 = 4ca;$$

eliminating $4c$, the equation of the curve is

$$y^2 = \frac{b^2}{a}x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The volume V of the cylinder of which the maximum is required is expressed by

$$V = \pi y^2(a - x),$$

or, by equation (1), $$V = \pi \frac{b^2}{a} x(a - x).$$

Hence we put $$f(x) = ax - x^2,$$

and the condition $f'(x) = 0$ gives

$$x = \tfrac{1}{2}a.$$

Consequently $a - x$, the altitude of the cylinder, is one half the altitude of the segment.

Examples XVII.

1\. Find the sides of the largest rectangle that can be inscribed in a semicircle of radius a. The sides are $a\sqrt{2}$ and $\frac{1}{2}a\sqrt{2}$.

2\. Determine the maximum right cone inscribed in a given sphere. The altitude is four thirds the radius of the sphere.

3\. Determine the maximum rectangle inscribed in a given segment of a parabola.
The altitude of the rectangle is two thirds that of the segment.

4\. Find the maximum cone of given slant height a.
The radius of the base is $\frac{1}{3}a\sqrt{6}$.

5\. A boatman 3 miles out at sea wishes to reach in the shortest time possible a point on the beach 5 miles from the nearest point of the shore; he can pull at the rate of 4 miles an hour, but can walk at the rate of 5 miles an hour; find the point at which he must land.

Express the whole time in terms of the distance of the required point from the nearest point of the shore.

He must land one mile from the point to be reached.

6. If a square piece of sheet-lead whose side is a have a square cut out at each corner, find the side of the latter square in order that the remainder may form a vessel of maximum capacity.

$V = (a-2x)^2 x$ The side of the square is $\frac{1}{6}a$.

7. A given weight is to be raised by means of a lever weighing n pounds per linear inch, which has its fulcrum at one end, and at a fixed distance a from the point of suspension of the weight w; find the length of the lever in order that the power required to raise the weight may be a minimum. $P(a+x) = wa + n\frac{(a+x)^2}{2}$ $\sqrt{\frac{2aw}{n}}$.

$\frac{dP}{dx} = 0$, $\therefore x+a = \sqrt{\frac{2wa}{n}}$

8. A rectangular court is to be built so as to contain a given area c^2, and a wall already constructed is available for one of the sides; find its dimensions so that the least expense may be incurred.

The side parallel to the wall is double each of the others.

9. Determine the maximum cylinder inscribed in a given cone. $C_y = \pi x^2 y$, $C = \frac{1}{3}\pi r^2 h$

The altitude of the cylinder is one third that of the cone. $\frac{x}{h-y} = \frac{r}{h}$, $\therefore y =$

10. Prove that the rectangle with given perimeter and maximum area is a square; also that the rectangle with given area and minimum perimeter is a square.

11. Find the side of the smallest square that can be inscribed in a square whose side is a.

Take as the independent variable the distance between the angles of the two squares. $y^2 = x^2 + (a-x)^2$, $x = \frac{a}{2}$, $y =$ $\frac{1}{2}a\sqrt{2}$.

12. Inscribe the maximum cone in a given paraboloid, the apex of the cone being at the middle point of the base of the paraboloid.

The altitude of the cone is half that of the paraboloid.

13. Find the maximum cylinder that can be inscribed in a sphere whose radius is a. The altitude is $\frac{2}{3}a\sqrt{3}$.

14. Through a point whose rectangular coordinates are a and b draw a line such that the triangle formed by this line and the coordinate axes shall be a minimum.

The intercepts on the axes are $2a$ and $2b$.

15. A high vertical wall is to be braced by a beam which must pass over a parallel wall a feet high and b feet distant from the other; find the length of the shortest beam that can be used for this purpose.

Take as the independent variable the inclination of the beam to the horizon. $y = \frac{a}{\sin\alpha} + \frac{b}{\cos\alpha}$, if $\tan^3\alpha = \frac{a}{b}$

$$(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}}.$$

16. From a point whose abscissa is c, on the axis of the parabola $y^2 = 4ax$, determine the shortest line to the curve.

The abscissa of the required point on the curve is $c - 2a$.

17. Determine the greatest rectangle that can be inscribed in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The sides are $a\sqrt{2}$ and $b\sqrt{2}$.

18. A cylinder is inscribed in a cone whose altitude is a, and the radius of whose base is b; determine the cylinder so that its total surface shall be a maximum, and thence show that there will be no maximum when $a < 2b$.

The altitude is $\dfrac{a^2 - 2ab}{2(a-b)}$.

19. Determine the cone of minimum volume described about a given sphere. The height is twice the diameter of the sphere.

20. A sphere has its centre in the surface of a given sphere whose radius is a; determine its radius in order that the area of the surface intercepted by the given sphere may be a maximum.

The area of a zone is measured by the product of the circumference of a great circle into the altitude of the zone.

$\frac{4}{3}a$.

21. Find the point, on the line joining the centres of two spheres whose radii are a and b, from which the greatest amount of spherical surface is visible.

The distance between the centres is divided in the ratio $a^{\frac{3}{2}} : b^{\frac{3}{2}}$.

22. In a given sphere, determine the inscribed cylinder whose entire surface is a maximum.

Solution :—

Using the notation of Art. 122, we find

$$f(x) = a^2 - x^2 + 2x\sqrt{(a^2 - x^2)};$$

whence $$f'(x) = -2x + 2\sqrt{(a^2 - x^2)} - \frac{2x^2}{\sqrt{(a^2 - x^2)}},$$

and $f'(x) = 0$ gives

$$x\sqrt{(a^2 - x^2)} = a^2 - 2x^2. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

Squaring, we have

$$5x^4 - 5a^2x^2 + a^4 = 0,$$

the roots of which are

$$x^2 = a^2(\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{\tfrac{1}{5}}):$$

but since the radical in equation (1) must be positive, we must have $x^2 < \frac{1}{2}a^2$; hence the altitude, $2x$, of the cylinder is

$$a\sqrt{(2 - \tfrac{2}{5}\sqrt{5})}.$$

23. In a given sphere determine the inscribed cone whose entire surface is a maximum.

The altitude of the cone is $\frac{a}{16}(23 - \sqrt{17})$.

24. A cylindrical trough is constructed by bending a given sheet of tin; its breadth being denoted by $2a$, find the radius of the cylinder when the capacity of the trough is a maximum.

Solution :—

Let x denote the radius of the cylinder; then $\frac{a}{x}$ will be the measure of the half-angle of the circular segment which constitutes a section of the trough. The area of the section will be expressed by

$$ax - x\cos\frac{a}{x} \cdot x\sin\frac{a}{x}. \qquad = ax - \frac{x^2}{2}\sin\frac{2a}{x}$$

Hence $f'(x) = 0$ gives

$$\cos\frac{a}{x}\left(a\cos\frac{a}{x} - x\sin\frac{a}{x}\right) = 0,$$

therefore $$\cos\frac{a}{x} = 0, \quad \text{or} \quad \tan\frac{a}{x} = \frac{a}{x}.$$

There is evidently a maximum between $\frac{a}{x} = 0$ and $\frac{a}{x} = \pi$. $\frac{a}{x} = \frac{\pi}{2}$ is a root of the first of the above equations, and since it is the only root of either equation between these limits, it must correspond to the maximum sought. Hence the section is a semicircle.

25. The illumination of a plane surface by a luminous point being directly as the cosine of the angle of incidence of the rays, and inversely as the square of its distance from the point; find the height at which a bracket-burner must be placed, in order that a point on the floor of a room at the horizontal distance a from the burner may receive the greatest possible amount of illumination.

The height is $\frac{a}{\sqrt{2}}$.

XVIII.

Methods of Discriminating between Maxima and Minima.

124. When the existence of a maximum or a minimum corresponding to a particular root a of the equation $f'(x) = 0$ is not obvious from the nature of the problem, it is necessary to determine whether $f'(x)$ *changes sign* as x passes through the value a.

If a change of sign does take place we have, in accordance with Art. 119, a *maximum* if, when x passes through the value a, the change of sign is from $+$ to $-$; that is, if $f'(x)$ is a *decreasing* function, and a *minimum* if the change of sign is from $-$ to $+$, in which case $f'(x)$ is an *increasing* function.

125. In many cases we are able to distinguish maxima from minima by examining the expression for $f'(x)$, as in the following examples.

Given $$f(x) = \frac{x}{\log x},$$

whence $$f'(x) = \frac{\log x - 1}{(\log x)^2};$$

$$f'(x) = 0 \quad \text{gives} \quad \log x = 1, \quad \text{or} \quad x = \epsilon.$$

Since $\log x$ is an increasing function, it is obvious that, as x increases through the value ϵ, $f'(x)$ increases; it therefore changes sign from $-$ to $+$, and consequently $f(\epsilon)$ is a minimum value of $f(x)$.

126. If $f'(x)$ does not change sign we have neither a maximum nor a minimum; thus, let

$$f(x) = x - \sin x,$$

whence $$f'(x) = 1 - \cos x.$$

In this case $f'(x)$ becomes zero when $x = 2n\pi$, n being zero or any integer, but does not change sign, since $1 - \cos x$ can never be negative; consequently $f(x)$ has neither maxima nor minima values, but is an *increasing* function for all values of x.

Alternate Maxima and Minima.

127. Let the curve

$$y = f(x)$$

be constructed, and suppose it to take the form represented in Fig. 12. There is a maximum value of $f(x)$ at B, another at D, and minima values occur at A, at C, and at E.

It is obvious that in a continuous portion of the curve maxima and minima ordinates must occur alternately, and must separate the curve into segments in which the ordinate is alternately an increasing and a decreasing function; hence, if $f(x)$ has maxi-

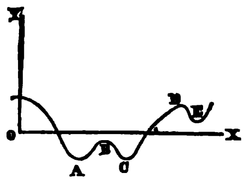


FIG. 12.

ma and minima values, they must occur alternately ***unless infinite values of the function intervene.*** It is also evident, with the same restriction, that a maximum is greater in value than either of the adjacent minima, but not necessarily greater than *any* other minimum; thus, in Fig. 12, the maximum at B is greater than the minima at A and C, but not greater than that at E.

128. As an illustration let us take the following function in which it is easy to discriminate between the maxima and minima values.

$$f(x) = x(x+a)^2(x-a)^3.$$

Whence,

$$f'(x) = (x+a)^2(x-a)^3 + 2x(x+a)(x-a)^3 + 3x(x+a)^2(x-a)^2,$$

$$= (x+a)(x-a)^2(6x^2 + ax - a^2).$$

a and $-a$ are evidently roots of $f'(x) = 0$; the roots derived by putting the last factor equal to zero and solving are $-\frac{1}{2}a$ and $\frac{1}{3}a$. Hence $f'(x)$ can be written in the form

$$f'(x) = 6(x+a)(x+\tfrac{1}{2}a)(x-\tfrac{1}{3}a)(x-a)^2,$$

in which the factors are so arranged that the corresponding roots are in order of magnitude.

When $x < -a$, $f'(x)$ is negative, and, if we regard x as increasing continuously, $f'(x)$ changes sign when $x = -a$, when $x = -\frac{1}{2}a$, and again when $x = \frac{1}{3}a$, but *not* when $x = a$.

Since $f'(x)$ is at first negative it changes sign from $-$ to $+$ when it first passes through zero, that is when $x = -a$; the corresponding value of $f(x)$ is therefore a minimum. Accordingly the value of $f(x)$ corresponding to the next root $x = -\frac{1}{2}a$ is a maximum, and that corresponding to $x = \frac{1}{3}a$ is another minimum; but there is neither a maximum nor a minimum corresponding to $x = a$.

129. When the function is continuous as in the above example, that is, does not become infinite for any finite value of x, it is always easy to determine by examining the function itself whether the last, or greatest value of x in question, gives a maximum or a minimum. Thus, in the above example, $f(x)$ evidently increases without limit as x increases without limit; therefore, the last value must be a minimum.

The Employment of a Substituted Function.

130. Since an increasing function of a variable increases and decreases with the variable, such a function will pass from a state of increase to a state of decrease, or the reverse, simultaneously with the variable; that is, it will reach a maximum or a minimum value at the same time with the variable.

This fact often enables us to simplify the determination of maxima and minima by substituting an increasing function of the given function for the given function itself. For example, if we have

$$f(x) = \sqrt{(b^2 + ax)} + \sqrt{(b^2 - ax)},$$

we may with advantage employ the square of the given function. The square is

$$2b^2 + 2\sqrt{(b^4 - a^2x^2)},$$

which is obviously a maximum when $x = 0$, and, since the square of a positive quantity is an increasing function, we infer that $f(x)$ is likewise a maximum for the same value of x.

131. It must however be observed that, if the original function assumes the value zero, the square will likewise become zero, and, since a square cannot become negative, this value (zero) will be a minimum value of the square even when the

original function has neither a maximum nor a minimum value. Thus let the given function be

$$\frac{x - 2a}{\sqrt{(x^2 - a^2)}}.$$

Employing the square, we put

$$f(x) = \frac{(x - 2a)^2}{x^2 - a^2},$$

whence

$$f'(x) = \frac{2(x^2 - a^2)(x - 2a) - 2x(x - 2a)^2}{(x^2 - a^2)^2} = \frac{2(x - 2a)(2ax - a^2)}{(x^2 - a^2)^2}.$$

$f'(x)$ changes sign when $x = 2a$ and when $x = \frac{1}{2}a$.

The former value of x gives $f(x) = 0$ which is a minimum of this function; but, when x passes through the value $2a$, the original function evidently changes sign, and is therefore neither a maximum nor a minimum. The other value of x, $\frac{1}{2}a$, gives an imaginary value to the original function; we therefore conclude that the given function has neither maxima nor minima values.

132. A *decreasing* function of the given function may also be employed; but, in this case, since the substituted function decreases with the increase of the given function and increases with its decrease, a maximum of the substituted function indicates a minimum, and a minimum indicates a maximum of the given function.

Thus, if we have

$$f(x) = \frac{x}{x^2 - 3x + 1},$$

the reciprocal may be employed. The reciprocal of this function is

$$\frac{x^2 - 3x + 1}{x} = x - 3 + \frac{1}{x};$$

whence, taking the derivative, we obtain

$$1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2},$$

which vanishes when $x = \pm 1$.

Since x^2 is an increasing function when x is positive, this derivative is evidently an *increasing* function when $x = 1$. The reciprocal is therefore a *minimum* for this value of x, and consequently $f(1)$ is a *maximum* value of $f(x)$. In a similar manner it may be shown that $f(-1)$ is a minimum.

Examples XVIII.

Determine the maxima and minima of the following functions:

1. $f(x) = x^x$. — A min. for $x = \dfrac{1}{\varepsilon}$.

2. $f(x) = \dfrac{\log x}{x^n}$. — A max. for $x = \varepsilon^{\frac{1}{n}}$.

3. $f(x) = \dfrac{(a - x)^3}{a - 2x}$. — A min. for $x = \tfrac{1}{4}a$.

4. $f(x) = \dfrac{1 + 3x}{\sqrt{(1 + 5x)}}$. — A min. for $x = -\tfrac{1}{15}$.

5. $f(x) = \sin 2x - x$. — A max. for $x = n\pi + \tfrac{1}{6}\pi$; a min. for $x = n\pi - \tfrac{1}{6}\pi$.

6. $f(x) = 2x^3 + 3x^2 - 36x + 12$. — A max. for $x = -3$; a min. for $x = 2$.

7. $f(x) = x^3 - 3x^2 - 9x + 5$. — A max. for $x = -1$; a min. for $x = 3$.

8. $f(x) = 3x^5 - 125x^3 + 2160x$. — A max. for $x = -4$ and $x = 3$; a min. for $x = -3$ and $x = 4$.

9. $f(x) = b + c(x - a)^{\frac{5}{3}}$. Neither a max. nor a min.

10. $f(x) = (x - 1)^4 (x + 2)^3$. A max. for $x = -\frac{5}{7}$;
a min. for $x = 1$.

11. $f(x) = (x - 9)^5 (x - 8)^4$. A max. for $x = 8$;
a min. for $x = 8\frac{4}{9}$.

12. Find the maximum and minimum ordinates of the curve

$$y = x^4 + 2x^3 - 12x^2 - 40x - 34.$$

A min. for $x = 2\frac{1}{2}$.

13. $f(x) = \dfrac{\sin^2 x}{\sqrt{(5 - 4\cos x)}}$.

To discriminate between maxima and minima observe that $f(0) = 0$ *and that* $f(x)$ *cannot become negative. See also Art.* 127.

A min. for $x = 0$;
a max. for $x = \cos^{-1} \frac{1}{2}(5 - \sqrt{13})$.

14. $f(x) = \dfrac{1 - x + x^2}{1 + x - x^2}$. A min. for $x = \frac{1}{2}$.

15. $f(x) = \dfrac{ax}{ax^2 - bx + a}$. *See Art.* 132. Max. for $x = 1$;
min. for $x = -1$.

16. $f(x) = \left(a^{x-1}\right)^{x+2}$.

Min. for $x = -\frac{1}{2}$ (a being positive).

17. $f(x) = (1 + x^{\frac{2}{3}})(7 - x)^{\frac{2}{3}}$.

Solve by putting $x = z^3$. *For method of discriminating between maxima and minima, see Art.* 129. Min. for $x = 0$, and $x = 7$;
max. for $x = 1$.

18. $f(x) = 5x^6 + 12x^5 - 15x^4 - 40x^3 + 15x^2 + 60x + 27$.
Min. for $x = -2$.

19. $f(x) = x^6 - 6x^4 + 4x^3 + 9x^2 - 12x + 3$.
Min. for $x = -2$, and $x = 1$;
max. for $x = -1$.

20. The top of a pedestal which sustains a statue a feet in height is b feet above the level of a man's eyes; find his horizontal distance from the pedestal when the statue subtends the greatest angle.

When the distance $= \sqrt{[b(a+b)]}$.

21. It is required to construct from two circular iron plates of radius a a buoy, composed of two equal cones having a common base, which shall have the greatest possible volume.

The radius of the base $= \frac{1}{3}a\sqrt{6}$.

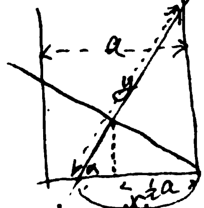
✓ 22. The lower corner of a leaf of a book is folded over so as just to reach the inner edge of the page; find when the crease thus formed is a minimum.

Solution :—

Let y denote the length of the crease, x the distance of the corner from the intersection of the crease with the lower edge, and a the width of the page.

By means of the relations of similar right triangles, the following expression is deduced:

$$y = \frac{x\sqrt{x}}{\sqrt{(x-\frac{1}{2}a)}}.$$



$x \cos\alpha \cdot \cos\alpha = x - \frac{1}{2}a$

$\therefore \cos\alpha = \sqrt{\frac{x-\frac{1}{2}a}{x}}$

subst. in:

$\frac{y}{x} = \frac{1}{\cos\alpha}$

Whence we obtain

$$x = \tfrac{3}{4}a,$$

which gives a minimum value of y.

✓ 23. Find when the area of the part folded over is a minimum.

When $x = \frac{2}{3}a$.

XIX.

The Employment of Derivatives Higher than the First.

133. To ascertain whether $f'(x)$ is an increasing or a decreasing function, (and thence whether $f(x)$ is a minimum or a maximum), it is frequently necessary to find the expression for its derivative, $f''(x)$. Now, if $f''(a)$ is found to have a *positive* value, it follows that $f'(x)$ is an *increasing* function when $x = a$,

and, as was shown in Art. 124, that $f(a)$ is a minimum. On the other hand, if we find that $f''(a)$ has a *negative* value, it follows that $f'(x)$ is a decreasing function, and that $f(a)$ is a maximum. To illustrate, let

$$f(x) = 3x^4 - 16x^3 - 6x^2 + 12,$$

then $$f'(x) = 12x^3 - 48x^2 - 12x.$$

The roots of $f'(x) = 0$ are $x = 0$, and $x = 2 \pm \sqrt{5}$.

In this case $f''(x) = 36x^2 - 96x - 12$,

hence $$f''(0) = -12;$$

$f(x)$ is therefore a *maximum* when $x = 0$.

It is unnecessary to find the values of $f''(x)$ for the other roots; for, since the function does not admit of infinite values, the maxima and minima occur alternately. The root $2 - \sqrt{5}$ being negative and $2 + \sqrt{5}$ positive, the root zero is intermediate in value, and therefore both the remaining roots give minima.

134. If $f'(x)$ contains a positive factor which cannot change sign, this factor may be omitted; since we can determine whether $f'(x)$ increases or decreases through zero by examining the sign of the derivative of the remaining factor. Thus, if

$$f(x) = \frac{x}{1 + x^2}, \qquad f'(x) = \frac{1 - x^2}{(1 + x^2)^2}.$$

Since $\dfrac{1}{(1 + x^2)^2}$ is always positive, we have only to determine whether the factor $1 - x^2$ changes sign. Denoting this factor by v, and putting $v = 0$, we have

$$x = \pm 1.$$

Now $$\frac{dv}{dx} = -2x$$

which is negative for $x = 1$ and positive for $x = -1$. These

roots, therefore, give respectively a maximum and a minimum value of $f(x)$.

135. There may be roots of the equation $f'(x) = 0$ which correspond to neither maxima nor minima, since it is a condition essential to the existence of such values that $f'(x)$ shall change sign. When such cases arise, the form assumed by the curve $y = f(x)$ in the immediate vicinity of the point at which $x = a$ will be one of those represented at A and B in Fig. 13.

At these points the value of $\tan\phi$ or $f'(x)$ is zero, but at A it is positive on both sides of the point, and $f(x)$ or y is an increasing function, while at B $f'(x)$ is negative on both sides of the point, and $f(x)$ is a decreasing function.

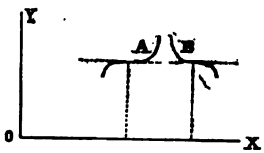


FIG. 13.

136. It is important to notice that at A the value zero assumed by $f'(x)$ constitutes a minimum value of this function, thus a root of $f'(x) = 0$ for which $f'(x)$ is a *minimum* corresponds to a case in which $f(x)$ is an *increasing* function. In like manner a root of $f'(x) = 0$ for which $f'(x)$ is a *maximum* is a case in which $f(x)$ is a *decreasing* function.

137. It follows from the preceding article and from Art. 124 that, if $f'(a) = 0$, then, of the two functions $f(x)$ and $f'(x)$, one will be a *maximum* and the other a *decreasing* function, or else one will be a *minimum* and the other an *increasing* function. Hence, if we consider the case in which the given function and several of its successive derivatives vanish for the same value of x, it is evident that when these functions are arranged in order *they will be either alternately maxima and decreasing functions, or alternately minima and increasing functions.*

138. Now suppose that $f''(x)$ is the first of these successive

derivatives that does not vanish when $x = a$, then, writing the series of functions

$$f(x), \quad f'(x), \quad f''(x), \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad f^{n-1}(x), \quad f^{n}(x),$$

let us assume first that $f^n(a)$ is *positive*. Then in the above series of functions $f^{n-1}(a)$, $f^{n-3}(a)$, etc., will be increasing functions while $f^{n-2}(a)$, $f^{n-4}(a)$, etc., will be minima.

Now whenever n is *odd*, the original function will belong to the first of these classes and will be an increasing function, while if n is *even* the original function will belong to the second class and will be a minimum.

On the other hand, if $f^n(a)$ has a *negative* value, the series of functions will be alternately decreasing functions and maxima; and when n is *odd* $f(a)$ will be a decreasing function, but when n is *even* $f(a)$ will be a maximum.

Thus we shall have neither maxima nor minima unless the first derivative, which does not vanish when $x = a$, is of an *even order;* but when this is the case we shall have a maximum or a minimum according as the value of this derivative is negative or positive.

139. The following function presents a case in which the above principle is advantageously employed.

$$f(x) = \varepsilon^x + \varepsilon^{-x} + 2\cos x,$$

$$f'(x) = \varepsilon^x - \varepsilon^{-x} - 2\sin x.$$

Zero is evidently a root of the equation $f'(x) = 0$.* In this case

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* Zero is the only root of $f'(x) = 0$ in this example; for

$$f''(x) = \frac{\varepsilon^{2x} - 2\varepsilon^x \cos x + 1}{\varepsilon^x} > \frac{(\varepsilon^x - 1)^2}{\varepsilon^x}.$$

$f''(x)$ therefore cannot be negative, hence $f'(x)$ cannot again assume the value zero.

$$f''(x) = \varepsilon^x + \varepsilon^{-x} - 2\cos x \quad \therefore \quad f''(0) = 0,$$

$$f'''(x) = \varepsilon^x - \varepsilon^{-x} + 2\sin x \quad \therefore \quad f'''(0) = 0,$$

$$f^{\text{iv}}(x) = \varepsilon^x + \varepsilon^{-x} + 2\cos x \quad \therefore \quad f^{\text{iv}}(0) = 4.$$

The fourth derivative being the first that does not vanish, and having a positive value, we conclude that $x = 0$ gives a minimum value of $f(x)$.

Examples XIX.

1. Show that $a\varepsilon^{kx} + b\varepsilon^{-kx}$ has a minimum value equal to $2\sqrt{(ab)}$.

Find the maxima and minima of the following functions:

2. $f(x) = x\sin x$.

A maximum for a value of x in the second quadrant satisfying the equation $\tan x = -x$.

3. $f(x) = \dfrac{a^2}{x} + \dfrac{b^2}{a - x}$.

The roots $x = \dfrac{a^2}{a+b}$ and $x = \dfrac{a^2}{a-b}$ give a min. and a max. if b is positive, but a max. and min. if b is negative.

4. $f(x) = 2\cos x + \sin^2 x$.

Solution:— $\qquad f'(x) = 2\sin x\,(\cos x - 1)$;

rejecting the factor $2(1 - \cos x)$, which is always positive, we put

$$v = -\sin x. \qquad \text{Hence } \frac{dv}{dx} = -\cos x.$$

A max. for $x = 2n\pi$;
a min. for $x = (2n + 1)\pi$.

5. $f(x) = \sin x(1 + \cos x)$.

A max. for $x = \frac{1}{3}\pi$;
a min. for $x = -\frac{1}{3}\pi$;
neither for $x = \pi$.

6. $f(x) = \dfrac{\sin x \cos x}{\cos^2(\frac{1}{3}\pi - x)}$.

Solution :—

On expanding $\cos^2(\frac{1}{3}\pi - x)$, the reciprocal reduces to

$$\tfrac{1}{4}(\cot x + 3 \tan x + 2\sqrt{3}).$$

Multiplying the derivative by $4 \sin^2 x$, we obtain

$$v = 3 \tan^2 x - 1\,;$$

whence $$\frac{dv}{dx} = 6 \tan x \sec^2 x.$$

Therefore $\tan x = \sqrt{\frac{1}{3}}$ gives a minimum, and $\tan x = -\sqrt{\frac{1}{3}}$ a maximum value of the reciprocal. Thus $x = \frac{1}{6}\pi$ gives to $f(x)$ a maximum value equal to $\frac{1}{3}\sqrt{3}$ and $x = -\frac{1}{6}\pi$, a minimum equal to $-\infty$.

7. $f(x) = \dfrac{x}{1 + x \tan x}$. A max. when $x = \cos x$.

8. $f(x) = \dfrac{x^2 + 4x + 10}{x^2 + 2x + 11}$. A min. for $x = -3$; a max. for $x = 4$.

9. $f(x) = \varepsilon^{\sin x \cos^2 x}$.

Maxima for $\cos x = \pm\sqrt{\frac{2}{3}}$; minimum for $\cos x = 0$ (the angles being taken in the first semicircle).

10. $f(x) = \sec x + \log \cos^2 x$.

Multiplying the derivative by $\cos^2 x$, *we obtain*

$$v = \sin x(1 - 2 \cos x).$$

A max. for $x = 0$, and $x = \pi$; a min. for $x = \pm\frac{1}{3}\pi$.

11. $f(x) = \dfrac{\tan^3 x}{\tan 3x}$. A min. for $x = 0$, $\frac{3}{8}\pi$, $\frac{5}{8}\pi$, and π; a max. for $x = \frac{1}{8}\pi$, $\frac{1}{2}\pi$, $\frac{7}{8}\pi$, etc.

12. $f(x) = \varepsilon^x + \varepsilon^{-x} - x^2$. A min. for $x = 0$.

13. $f(x) = 4x^2 + \cos 2x - \frac{1}{2}(\varepsilon^{2x} + \varepsilon^{-2x})$. A max. for $x = 0$.

14. $f(x) = (3 - x)\varepsilon^{2x} - 4x\varepsilon^{x} - x$. Is there a maximum or a minimum corresponding to $x = 0$? Neither.

XX.

Implicit Functions.

140. Let the relation between the variables x and y be expressed by the equation

$$F(x, y) = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and let it be required to find the maxima and minima values of either variable. The value of the derivative $\dfrac{dy}{dx}$ may be found as in Art. 66, in the form

$$\frac{dy}{dx} = \frac{u}{v}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which u and v are in general functions of x and y.

If equation (1) be regarded as the equation of a curve, it is evident that wherever the tangent to the curve is parallel to the axis of x we shall have

$$u = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Hence the values of x and y which satisfy simultaneously equations (1) and (3), and *do not make* $v = 0$, will give points at which the ordinate is a maximum or a minimum according as the value of the second derivative is negative or positive. The corresponding value of the second derivative is most readily obtained by the formula deduced in the next article.

141. From equation (2) we obtain

$$\frac{d^2y}{dx^2} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In the case which now presents itself we have $u = 0$, which causes the second term of the numerator to vanish, and moreover, since $\dfrac{dy}{dx} = 0$, the value of $\dfrac{du}{dx}$ is that which would be obtained by taking the derivative of u on the supposition that y is constant. Hence, indicating by brackets the particular values which the derivatives take when $\dfrac{dy}{dx} = 0$, we have

$$\left[\frac{d^2y}{dx^2}\right] = \frac{\left[\dfrac{du}{dx}\right]}{v}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

in which the values of x and y determined by equations (1) and (3) are to be substituted.

142. To illustrate, let

$$xy^2 - x^2y = 2a^3, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which a denotes a positive constant. Differentiating

$$y^2 + 2xy\frac{dy}{dx} - 2xy - x^2\frac{dy}{dx} = 0;$$

therefore,

$$\frac{dy}{dx} = \frac{y(2x-y)}{x(2y-x)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In this example $u = y(2x - y)$ and $v = x(2y - x)$; putting $u = 0$, we obtain

$$y = 0 \qquad \text{or} \qquad y = 2x.$$

Substituting $y = 0$ in (1) gives an infinite value to x, but $y = 2x$ gives

$$x = a \quad \text{and} \quad y = 2a.$$

Formula (5) of the preceding article gives

$$\left[\frac{d^2y}{dx^2}\right] = \frac{2y}{x(2y - x)};$$

substituting $x = a$ and $y = 2a$, we find

$$\left[\frac{d^2y}{dx^2}\right]_{a,\,2a} = \frac{4}{3a}$$

which is positive; hence $2a$ is a minimum value of y.

In like manner maxima and minima values of x may be found by employing the condition

$$v = 0,$$

and a formula, similar to (5) of Art. 141, to discriminate between the maxima and minima.

143. Fig. 14 represents the curve corresponding to equation (1) of the preceding article; viz.,

$$xy^2 - x^2y = 2a^3. \quad . \quad . \quad . \quad (1)$$

The ordinate at A is the minimum found above, and the abscissa of B $(-2a, -a)$ is a maximum.

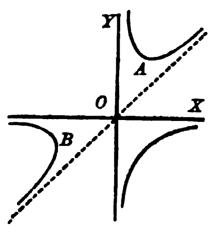

FIG. 14.

Infinite Values of the Derivative.

144. If y denotes a function of x, a maximum or a minimum of y occurs whenever the derivative $\dfrac{dy}{dx}$ changes sign as x

passes through a certain value. Since a function may change sign on passing through infinity, it is necessary to consider the cases in which

$$\frac{dy}{dx} = \infty.$$

But, since in such cases

$$\frac{dx}{dy} = 0,$$

this condition usually corresponds to a point like B in Fig. 14 at which x, instead of y, is a maximum or a minimum. In such cases x does not *pass through* the value which makes the derivative infinite, and therefore the condition necessary for a maximum or minimum value of y is not fulfilled. The function

$$y = (x - a)^{\frac{1}{4}}$$

presents a case of this kind; for

$$\frac{dy}{dx} = \frac{1}{4(x - a)^{\frac{3}{4}}},$$

which is infinite for $x = a$, but, since y becomes imaginary, x cannot pass through the value a; an essential condition for a maximum or a minimum value of y is therefore wanting, but it is easily shown that a minimum abscissa of the curve $y^4 = x - a$ occurs at the point in question $(a, 0)$.

145. It is noteworthy also that since by Art. 104 $f'(x)$ is infinite whenever $f(x)$ is infinite for a finite value of x, the roots of the equation

$$f'(x) = \infty$$

will include the values of x which render $f(x)$ infinite whenever such values exist. Thus, if

$$f(x) = \frac{a^2 x}{(x - a)^2},$$

$$f'(x) = -\frac{a^2(x + a)}{(x - a)^3}.$$

$x = a$ makes $f'(x)$ infinite, but it also makes $f(x)$ infinite.*

Putting $f'(x) = 0$, we have $x = -a$, which corresponds to a minimum value of $f(x)$, since, for this value of x, $f'(x)$ is an increasing function.

146. If the equation

$$f'(x) = \infty$$

has a root, for which $f(x)$ remains finite and does not become imaginary, it is still necessary to ascertain whether $f'(x)$ changes sign, since there will be a maximum if the change be from $+$ to $-$, and a minimum if it be from $-$ to $+$.

The form of the curve

$$y = f(x)$$

in the vicinity of a maximum or a minimum ordinate of this variety is represented at A and B in Fig. 15.

As an example, let

$$f(x) = (x^{\frac{1}{3}} - b^{\frac{1}{3}})^{\frac{2}{3}},$$

whence
$$f'(x) = \tfrac{2}{9}x^{-\frac{2}{3}}(x^{\frac{1}{3}} - b^{\frac{1}{3}})^{-\frac{1}{3}}.$$

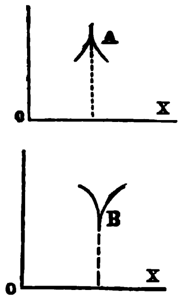

FIG. 15.

$f'(x)$ is infinite when $x = 0$ and when $x = b$. When $x = 0$ $f'(x)$ does not change sign, since $x^{-\frac{2}{3}}$ cannot be negative, but when $x = b$ it changes sign from $-$ to $+$; hence $f(x)$ has a minimum value when $x = b$.

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* When as in this example $f'(x)$ *changes sign* as x passes through the value that makes $f(x)$ infinite, this value of $f(x)$ may be regarded as an infinite maximum. See the corresponding curve Fig. 17, Art. 202.

Examples XX.

1. Given $25y^2 - 6xy + x^2 - 9 = 0$, to find the maxima and minima of y. Min. for $x = -\frac{9}{4}$; max. for $x = \frac{9}{4}$.

2. Given $x^4 + 2ax^2y - ay^3 = 0$, to find the maxima and minima of y. Min. for $x = \pm a$.

3. Given $y^2 - x^2y + x - x^3 = 0$, to prove that $x = -1$ gives a maximum value of y.

4. Given $3a^2y^2 + xy^3 + 4ax^3 = 0$. Show that when $x = \frac{3}{2}a$, y has a maximum value, namely $-3a$, the value of $\left[\frac{d^2y}{dx^2}\right]$ being then $-\frac{8}{5a}$.

5. Given $y^2 + 2x^2y + 4x - 3$, to prove that $x = 1$ gives a maximum value of y.

6. Given $y^3 + x^3 - 3axy = 0$. A max. for $x = a\sqrt[3]{2}$; a min. for $x = 0$.

7. Find maxima and minima of the following functions:

$f(x) = (x^{\frac{2}{3}} - b^{\frac{2}{3}})^{\frac{1}{2}}$. A min. for $x = 0$.

8. $f(x) = (x^2 - b^2)^{\frac{2}{3}}$. A max. for $x = 0$; a min. for $x = \pm b$.

9. $f(x) = (x^2 + 3x + 2)^{\frac{2}{3}} + x^{\frac{2}{3}}$.

$f'(x) = \infty$ gives min. corresponding to $x = -2$, $x = -1$ and $x = 0$.
$f'(x) = 0$ gives two intermediate maxima.

10. $f(x) = (x^2 + 2x)^{\frac{2}{3}} - (x + 3)^{\frac{4}{3}}$. Max. for $x = \frac{1}{4}(-3 \pm \sqrt{17})$; min. for $x = 0$ and $x = -2$.

11. $f(x) = \frac{(x + 3)^3}{(x + 2)^2}$. *See Art.* 145. A min. for $x = 0$.

12. $f(x) = (x-a)^{\frac{4}{5}}(x-b)^{\frac{2}{5}} + c.$ A max. for $x = \dfrac{2b+a}{3}$;
min. for $x = a$ and $x = b$.

13. $f(x) = \dfrac{(x-a)(x-b)}{x^2}.$ A min. for $x = \dfrac{2ab}{a+b}$.

14. $f(x) = (x-a)^{\frac{2}{3}}(x-b)^{\frac{1}{3}}.$

Solutions for $x = a$ and $x = \frac{1}{3}(2b+a)$; if $b > a$, the former gives a max. and the latter a min.

15. $f(x) = \log\cos x - \cos x.$ Max. for $x = 0$.

16. $f(x) = (x-1)^3(x+1)^{-2}.$ Max. for $x = -5$.

Miscellaneous Examples.

1. $f(x) = \dfrac{x-1}{x^3 - 3x^2 + 2x + 54}.$ *Use the reciprocal.*
Max. for $x = 4$.

2. $f(x) = \dfrac{x^2 - x + 1}{x^2 + x - 1}.$ A max. for $x = 0$; a min. for $x = 2$.

3. $f(x) = x^{-a}\varepsilon^{bx}.$ A min. for $x = \dfrac{a}{b}$.

4. The equation of the path of a projectile being

$$y = x\tan\alpha - \frac{x^2}{4h\cos^2\alpha},$$

find the value of x when y is a maximum; also the maximum value of y. Max. when $x = h\sin 2\alpha$, and $y = h\sin^2\alpha$.

5. In a given sphere inscribe the greatest rectangular parallelopiped.

Solution :—

Regarding any one edge as of fixed length, it is easy to show that the other two edges are equal. Hence the three edges are equal.

6. In a given cone inscribe the greatest rectangular parallelopiped.

Solution :—

Regarding the parallelopiped as inscribed in a cylinder which is itself inscribed in the cone, the base is evidently a square, and the altitude is that of the maximum cylinder. See Ex. XVII, 9.

7. Given
$$u = a\cos^2 x + b\cos^2 y, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$
x and y being connected by the equation
$$y - x = \tfrac{1}{4}\pi; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$
find the values of x when u is a maximum or a minimum.

Solution :—
$$\frac{du}{dx} = -a\sin 2x - b\sin 2y\,\frac{dy}{dx}.$$

From the second equation we have
$$\frac{dy}{dx} = 1, \qquad \text{and} \qquad 2y = 2x + \tfrac{1}{2}\pi;$$

whence
$$\sin 2y = \cos 2x;$$

therefore
$$\frac{du}{dx} = -a\sin 2x - b\cos 2x = 0,$$

and
$$x = \tfrac{1}{2}\tan^{-1}\left(-\frac{b}{a}\right).$$

This equation has a root in each quadrant, and, as u does not admit of infinite values, these roots correspond to alternate maxima and minima. Since $\left.\dfrac{du}{dx}\right]_0 = -b$, u is a decreasing function when $x = 0$; therefore the value in the first quadrant gives a minimum.

8. Given one angle A of a right spherical triangle, to find when the difference between the sides which contain it is a maximum.

We have in this case $\tan c \cos A = \tan b$, *in which* b *and* c *are the variables and, since* $c - b$ *is a maximum,* $\frac{dc}{db} = 1$.

A max. when $b = \tan^{-1}(\sqrt{\cos A})$.

9. A Norman window consists of a rectangle surmounted by a semicircle. Given the perimeter, required the height and breadth of the window when the quantity of light admitted is a maximum.

The radius of the semicircle is equal to the height of the rectangle.

10. A tinsmith was ordered to make an open cylindrical vessel of given volume, which should be as light as possible; find the ratio between the height and the radius of the base.

The height equals the radius of the base.

11. What should be the ratio between the diameter of the base and the height of cylindrical fruit-cans in order that the amount of tin used in constructing them may be the least possible?

The height should equal the diameter of the base.

12. Determine the circle having its centre on the circumference of a given circle so that the arc included in the given circle shall be a maximum.

A max. for the value of θ which is in the first quadrant.

13. Given the vertical angle of a triangle and its area; find when its base is a minimum. The triangle is isosceles.

14. Prove that, of all circular sectors of the same perimeter, the sector of greatest area is that in which the circular arc is double the radius.

15. Find the minimum isosceles triangle circumscribed about a parabolic segment.

The altitude of the triangle is four-thirds the altitude of the segment.

16. Find the least isosceles triangle that can be described about a given ellipse, having its base parallel to the major axis.

The height is three times the minor semi-axis.

17. Inscribe the greatest parabolic segment in a given isosceles triangle.

The altitude of the segment is three-fourths that of the triangle.

18. A steamer whose speed is 8 knots per hour and course due north sights another steamer directly ahead, whose speed is 10 knots, and whose course is due west. What must be the course of the first steamer to cross the track of the second at the least possible distance from her?

N. $53^\circ\ 8'$ *W.*

19. Determine the angle which a rudder makes with the keel of a ship when its turning effect is the greatest possible.

Solution :—

Let ϕ denote the angle between the rudder and the prolongation of the keel of the ship; then if b is the area of the rudder that of the stream of water intercepted will be $b \sin \phi$: the resulting force being decomposed, the component perpendicular to the rudder contains the factor $\sin^2 \phi$. Again decomposing this force, and taking the component that is perpendicular to the keel of the ship, which is the only part of the original force that is effective in turning the ship, the expression to be made a maximum is

$$\sin^2 \phi \cos \phi.$$

Whence we obtain

$$\tan \phi = \sqrt{2}.$$

✓20. The work of driving a steamer through the water being proportional to the cube of her speed, find her most economical rate per hour against a current running a knots per hour. The wk per knot should be a min.

Solution :—

Let v denote the speed of the steamer in knots per hour. The work per hour will then be denoted by kv^3, k being a constant, and the actual distance the steamer advances per hour by $v - a$. The work per knot made good is therefore expressed by

$$\frac{kv^3}{v - a}.$$

Whence we obtain the result

$$v = \tfrac{3}{8}a.$$

21. To find the parallel on the earth's surface at which the difference between the geographical and the geocentric latitude is greatest.

Solution :—

Assuming the meridian to be an ellipse, and taking the origin at the centre, its equation will be

$$y = \frac{b}{a}\sqrt{(a^2 - x^2)}.$$

Let θ denote the geocentric latitude, and ψ the geographical latitude; then

$$\tan(\psi - \theta) = \frac{\tan\psi - \tan\theta}{1 + \tan\psi\tan\theta}$$

is to be a maximum. $\tan\theta = \dfrac{y}{x}$, and, since $\psi = \phi - \frac{1}{2}\pi$,

$$\tan\psi = -\cot\phi = -\frac{dx}{dy}.$$

Substituting these values in the expression for $\tan(\psi - \theta)$ we obtain $x = \dfrac{a}{\sqrt{2}}$ which gives the maximum required.

22. The position of a point in each of two media (A and B) separated by a plane surface, being given, it is required to find the path described by a particle which goes from one point to the other in the shortest possible time, the velocity of the particle being constant in each medium.

Solution :—

The path is evidently composed of two straight lines, one in each medium. It is easily proved that the plane passing through the two points, and normal to the separating surface, contains these lines. For, if not, let the path be projected on this plane, then the portion of the

new path thus formed, contained in each medium, will be shorter than the corresponding portion of the original path; therefore the new path will be described in less time than the original path.

Let a and b denote the lengths of the perpendiculars let fall from the given points upon the separating surface, u the velocity in the medium A, and v that in the medium B.

Let the particle be supposed to move from A to B, and let i denote the angle of incidence, that is, the angle between a and the path in A, and r the corresponding angle in the medium B.

If c denotes the projection of the path upon the separating surface, which is constant, the relation between i and r is expressed by

$$a \tan i + b \tan r = c. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The time occupied in describing the path is

$$\frac{a \sec i}{u} + \frac{b \sec r}{v}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

putting the differential of this expression equal to zero, we have

$$\frac{a}{u} \sec i \tan i \, di = -\frac{b}{v} \sec r \tan r \, dr. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From (1) we obtain

$$a \sec^2 i \, di = -b \sec^2 r \, dr; \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

dividing (3) by (4) to eliminate di and dr, we have

$$\frac{1}{u} \cdot \frac{\tan i}{\sec i} = \frac{1}{v} \cdot \frac{\tan r}{\sec r},$$

or

$$\frac{v}{u} = \frac{\sin r}{\sin i}.$$

The resulting path is that actually described by a refracted ray of light, if we suppose the ratio of the velocities to equal the index of refraction.

This problem was originally proposed by Fermat.

23. The orbits of Venus and the Earth being regarded as circular, determine the distance between the planets when the brightness of Venus is a maximum.

Solution :—

Let r denote the required distance, a the radius of the Earth's orbit, b the radius of the orbit of Venus, and ϕ the angle between b and r.

Denoting the apparent semi-diameter of the disk at the distance unity by c, at the distance r it will be denoted by $\frac{c}{r}$, the corresponding area of the entire disk by $\frac{\pi c^2}{r^2}$, and the illuminated portion of the disk by

$$\frac{\pi c^2}{r^2} \cdot \frac{1 + \cos\phi}{2}.$$

We therefore require the maximum value of

$$\frac{1 + \cos\phi}{r^2};$$

hence, eliminating $\cos\phi$ by the relation

$$a^2 = b^2 + r^2 - 2br\cos\phi,$$

we obtain
$$f(r) = \frac{b^2 - a^2 + r^2 + 2br}{r^3}, \quad \ldots \ldots \quad (1)$$

and, solving the equation $f'(r) = 0$, we derive

$$r = \sqrt{(b^2 + 3a^2)} - 2b, \quad \ldots \ldots \quad (2)$$

the other root being inadmissible since it is always negative.

If we regard the brightness as a function of the time, the equation for determining the maxima and minima of $f(r)$ is

$$f'(r)\frac{dr}{dt} = 0.$$

Hence the roots of the equation

$$\frac{dr}{dt} = 0$$

are also solutions of the problem. These roots evidently correspond to the maxima and minima values of r, which occur at the conjunctions; at each of which the brightness is obviously a minimum. Hence at the points determined by equation (2) it is a maximum.

Taking the mean distance a of the Earth from the sun as unity, that of Venus according to G. W. Hill is 0.723; substituting these values for a and b in the above expression for r, we obtain $r = 0.431$, the corresponding elongation (angle between a and r) will be found to be $39° 43'$.

CHAPTER VII.

The Development of Functions in Series.

XXI.

The Nature of an Infinite Series.

147. A FUNCTION which can be expressed by means of a limited number of integral terms, involving powers of the independent variable with positive integral exponents only, is called a *rational integral function.*

When $f(x)$ is *not* a rational integral function, it is usually possible to derive an unlimited series of terms rational and integral with respect to x, which may be regarded as an algebraic equivalent for the function. The process of deriving this series is called the *development* of the function into an *infinite series.*

When the given function is in the form of a rational fraction, the ordinary process of division (the dividend and divisor being arranged according to ascending powers of x) suffices to effect the development. Thus—

$$\frac{1+x}{1-x} = 1 + 2x + 2x^2 + 2x^3 + \cdots,$$

a series of terms arranged according to ascending powers of x, each coefficient after the absolute term being 2.

It is to be observed, in the first place, that, owing to the indefinite number of terms in the second member, the equation as written above cannot be verified numerically for an assumed value of x. In this case, however, the process not

only gives us the series, but the remainder after any number of terms. Thus carrying the quotient to the term containing x^n, and writing the remainder, we have

$$\frac{1+x}{1-x} = 1 + 2x + 2x^2 \cdots + 2x^n + \frac{2x^{n+1}}{1-x}.$$

This equation may now be verified numerically for any assumed value of x; or algebraically by multiplying each member by $1 - x$, thus obtaining an identity.

The ordinary process of extracting the square root of a polynomial furnishes an example of a series which may be extended so as to include as many terms as we please; but this process gives us no expression for the remainder.

148. Assuming that $f(x)$ admits of development into a series involving ascending powers of x, and denoting the remainder after $n + 1$ terms by R, we may write

$$f(x) = A + Bx + Cx^2 + \cdots + Nx^n + R, \quad . \quad . \quad . \quad (1)$$

in which $A, B, C, \ldots N$ denote coefficients independent of x, and as yet unknown; the value of R is however not independent of x. If the coefficients $B, C, \ldots N$ admit of finite values, it may be assumed that R is a function of x which vanishes when $x = 0$; and in accordance with this assumption equation (1) becomes, when $x = 0$,

$$f(0) = A, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which determines the first term of the series. If in any case the value of $f(0)$ is found to be infinite, we infer that the proposed development is impossible.

149. When the coefficients $B, C, \ldots N$ admit of finite values, and the value of the function to be developed remains

finite, R will have a finite value. If moreover the value of R decreases as n increases, and can be made as small as we please, by sufficiently increasing n, the series is said to be *convergent*, and may be employed in finding an approximate value of the function $f(x)$; the closeness of the approximation increasing with the number of terms used. A series in which R does not decrease as n increases is said to be *divergent*.

When the successive terms of a series decrease it does not necessarily follow that the series is convergent; for the value of the equivalent function, and consequently that of R, may be infinite. To illustrate, if we put $x = 1$ in the series

$$x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \cdots,$$

we obtain the numerical series

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \cdots;$$

it can be shown that, by taking a sufficient number of terms, the sum of this series may be made to exceed any finite limit, *the value of the equivalent or generating function of the above series being in fact infinite when* $x = 1$.*

150. Since R vanishes with x, every series for which finite coefficients can be determined is convergent for certain small values of x. In some cases there are limiting values of x, both positive and negative, within which the series is convergent, while for values of x without these limits the series is divergent. These values of x are called *the limits of convergence.*

---

* If we consider the first two terms separately, and regard the other terms as arranged in groups of two, four, eight, sixteen, etc., the groups will end with the terms $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, etc. The sum of the fractions in the first group exceeds $\frac{2}{4}$ or $\frac{1}{2}$, the sum of those in the second exceeds $\frac{4}{8}$ or $\frac{1}{2}$, and so on; hence the sum of $2N$ such groups exceeds the number N, and N may be taken as large as we choose.

The generating function in this case is $\log \dfrac{1}{1-x}$, and unity is the limit of convergence.

We shall now demonstrate a theorem by which a function in the form $f(x_0+h)$ may be developed into a series involving powers of h, and in Section XXII we shall show how this theorem is transformed so as to give the expansion of $f(x)$ in powers of x.

Taylor's Theorem.

151. A function of h of the form $f(x_0+h)$ in general admits of development in a series involving ascending powers of h. We therefore assume

$$f(x_0+h)=A_0+B_0h+C_0h^2+\cdot\cdot\cdot+N_0h^n+R_0, \quad . \quad . \quad (1)$$

in which A_0, B_0, C_0, . . . N_0 are independent of h, while R_0 is a function of h which vanishes when h is zero. Hence, making $h=0$, we have

$$f(x_0)=A_0.$$

We have now to find the values of B_0, C_0, . . . N_0, which are evidently functions of x_0. For this purpose we put

$$x_1=x_0+h, \qquad \text{whence} \qquad h=x_1-x_0;$$

substituting, equation (1) takes the form

$$f(x_1)=f(x_0)+B_0(x_1-x_0)+C_0(x_1-x_0)^2\cdot\cdot\cdot+N_0(x_1-x_0)^n+R_0,$$

in which we may regard x_1 as constant and x_0 as variable. Replacing the latter by x, and its functions, B_0, C_0, . . . N_0, and R_0, by B, C, . . . N, and R, we have

$$f(x_1)=f(x)+B(x_1-x)+C(x_1-x)^2\cdot\cdot\cdot+N(x_1-x)^n+R. \quad . \quad (2)$$

Taking derivatives with respect to x, we have

$$0 = f'(x) - B + (x_1 - x)\frac{dB}{dx} - 2C(x_1 - x) + (x_1 - x)^2\frac{dC}{dx} \cdots$$

$$- nN(x_1 - x)^{n-1} + (x_1 - x)^n\frac{dN}{dx} + \frac{dR}{dx}. \quad . \quad . \quad . \quad (3)$$

To render the development possible, B, C, ... N, and R must have such values as will make equation (3) *identical*, that is, true for all values of x.

152. It is evident that B may be so taken as to cause the first two terms of equation (3) to vanish, and that, this being done, C can be so determined as to cause the coefficient of $(x_1 - x)$ to vanish, D so as to make the coefficient of $(x_1 - x)^2$ vanish, and so on. The requisite conditions are

$$f'(x) - B = 0, \qquad \frac{dB}{dx} - 2C = 0, \qquad \frac{dC}{dx} - 3D = 0, \text{ etc.,}$$

and finally $$(x_1 - x)^n\frac{dN}{dx} + \frac{dR}{dx} = 0.$$

From these conditions we derive

$$B = f'(x), \qquad C = \tfrac{1}{2}\frac{dB}{dx} = \tfrac{1}{2}f''(x),$$

$$D = \tfrac{1}{3}\frac{dC}{dx} = \frac{1}{1\cdot 2\cdot 3}f'''(x), \qquad E = \frac{1}{1\cdot 2\cdot 3\cdot 4}f^{\text{iv}}(x),$$

and in general $$N = \frac{1}{1\cdot 2\cdots n}f^n(x).$$

Putting x_0 for x, and substituting in equation (1) the values of A_0, B_0, C_0, ... N_0, we obtain

$$f(x_0+h) = f(x_0) + f'(x_0)h + f''(x_0)\frac{h^2}{1\cdot 2}\cdots + f^n(x_0)\frac{h^n}{1\cdot 2\cdots n} + R_0. \quad (4)$$

This result is called Taylor's Theorem, from the name of its discoverer, Dr. Brook Taylor, who first published it in 1715.

It is evident from equation (4) that *the proposed expansion is impossible* when the given function or any of its derived functions is infinite for the value x_0.

Lagrange's Expression for the Remainder.

153. R denotes a function of x which takes the value R_0 when $x = x_0$, and becomes zero when $x = x_1$. It has been shown in the preceding article that R must also satisfy the equation

$$(x_1 - x)^n \frac{dN}{dx} + \frac{dR}{dx} = 0,$$

or, substituting the value of N determined above,

$$\frac{dR}{dx} = -\frac{(x_1 - x)^n}{1 \cdot 2 \cdots n} f^{n+1}(x). \quad . \quad . \quad . \quad . \quad . \quad (5)$$

This equation shows that $\frac{dR}{dx}$ cannot become infinite for any value of x between x_0 and x_1, provided $f^{n+1}(x)$ remains finite and real while x varies between these limits. Since it follows from the theorem proved in Art. 104 that all preceding derivatives must be likewise finite, the above hypothesis is equivalent to the assumption that $f(x)$ *and its successive derivatives to the* $(n + 1)$*th inclusive remain finite and real while* x *varies from* x_0 *to* $x_0 + h$.

154. Let P denote any assumed function of x which, like R, takes the value R_0 when $x = x_0$ and the value zero when $x = x_1$, and whose derivative $\frac{dP}{dx}$ does not become infinite or imaginary for any value of x between these limits.

Then, R_0 *being assumed to be finite*, $P - R$ denotes a function of x which vanishes both when $x = x_0$ and when $x = x_1$ and whose derivative cannot become infinite for any intermediate value of x. It follows therefore that the value of this function cannot become infinite for any intermediate value of x.

Since, as x varies from x_0 to x_1, $P - R$ starts from the value zero and returns to zero again, without passing through infinity, its numerical value must pass through a maximum; hence its derivative cannot retain the same sign throughout, and as it cannot become infinite it must necessarily become zero for some intermediate value of x. Since $x_1 = x_0 + h$ this intermediate value of x can be expressed by $x_0 + \theta h$, θ being a *positive proper fraction*. It is therefore evident that at least one value of x of the form

$$x = x_0 + \theta h$$

will satisfy the equation

$$\frac{dP}{dx} - \frac{dR}{dx} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

155. The value of P will fulfil the required conditions if we assume

$$P = \frac{(x_1 - x)^{n+1}}{h^{n+1}} \cdot R_0,$$

for this function takes the value R_0 when $x = x_0$ and vanishes when $x = x_1$; moreover its derivative with reference to x, viz.,

$$\frac{dP}{dx} = -\frac{(n+1)(x_1 - x)^n}{h^{n+1}} \cdot R_0, \quad . \quad . \quad . \quad . \quad (7)$$

does not become infinite for any intermediate value of x. Substituting in equation (6) the values of the derivatives given in equations (5) and (7), and solving for R_0, we obtain

$$R_0 = \frac{h^{n+1}}{1 \cdot 2 \cdots n \cdot (n+1)} f^{n+1}(x_0 + \theta h). \quad . \quad . \quad . \quad (8)$$

This expression for the remainder was first given by Lagrange.

The series may now be written thus:

$$f(x_0+h)=f(x_0)+f'(x_0)h+f''(x_0)\frac{h^2}{1\cdot 2}\cdots$$
$$+f^n(x_0)\frac{h^n}{1\cdot 2\cdots n}+f^{n+1}(x_0+\theta h)\frac{h^{n+1}}{1\cdot 2\cdots(n+1)}\cdot\cdot\quad\cdot\quad\cdot\ (9)$$

It should be noticed that the above expression for the remainder after $n+1$ terms differs from the next, or $(n+2)$th term of the series, simply by the addition of θh to x_0.

A Second Expression for the Remainder.

156. The function P will also fulfil the conditions specified in Art. 154 when assumed in the form *

$$P=R_0\frac{f^n(x_1)-f^n(x)}{f^n(x_1)-f^n(x_0)},$$

since this function becomes equal to R_0 when $x=x_0$, and is zero when $x=x_1$; and moreover its derivative; viz.,

$$\frac{dP}{dx}=-R_0\frac{f^{n+1}(x)}{f^n(x_1)-f^n(x_0)},$$

* The form

$$P=R_0\frac{\varphi(x_1)-\varphi(x)}{\varphi(x_1)-\varphi(x_0)},$$

in which φ is continuous between $\varphi(x_1)$ and $\varphi(x_0)$, includes all the forms in which P can be assumed. The resulting value of R_0 is

$$R_0=\theta[\varphi(x_1)-\varphi(x_0)]\frac{f^{n+1}(x_0+\theta h)}{\varphi'(x_0+\theta h)}\cdot\frac{h^n}{1\cdot 2\cdots n},$$

which is accordingly the most general form of R_0 that can be derived by this method, and includes that of Cauchy as well as other forms of the remainder. See *The Mathematical Messenger* for April, 1873, in which this method of proving Taylor's Theorem was first published.

is continuous between the limits $x = x_0$ and $x = x_1$, because $f^{n+1}(x)$ is, by the hypothesis of Art. 153, continuous between these limits.

Employing this expression for $\dfrac{dP}{dx}$ in place of that given in equation (7), we obtain

$$\frac{R_0}{f^n(x_1) - f^n(x_0)} = \frac{(x_1 - x_0 - \theta h)^n}{1 \cdot 2 \cdots n} = \frac{h^n(1-\theta)^n}{1 \cdot 2 \cdots n}.$$

Since $(1-\theta)^n$, is a positive proper fraction, it may for simplicity be denoted by θ; hence we have

$$R_0 = \theta[f^n(x_0 + h) - f^n(x_0)] \frac{h^n}{1 \cdot 2 \cdots n}. \quad . \quad . \quad (10)$$

Since this value of R_0 falls between

$$0 \quad \text{and} \quad [f^n(x_0 + h) - f^n(x_0)] \frac{h^n}{1 \cdot 2 \cdots n},$$

the value of $f(x_0 + h)$ is intermediate between the two expressions,

$$f(x_0) + f'(x_0)h \cdots + f^n(x_0)\frac{h^n}{1 \cdot 2 \cdots n},$$

and $$f(x_0) + f'(x_0)h \cdots + f^n(x_0 + h)\frac{h^n}{1 \cdot 2 \cdots n}.$$

It is to be noticed that the latter differs from the former only in the substitution of $x_0 + h$ for x_0 in the last term.

Limits to the Application of Taylor's Theorem.

157. When, for a given value of h, the hypothesis stated in Art. 153 is true for every value of n, the expressions for R_0, and

consequently the complete form of Taylor's Theorem expressed in equation (9), Art. 155, are applicable for all values of n.

In many cases, however, when the value of x_0 is given, there exist limiting values of h, either positive or negative, such that for numerically greater values of h equation (9) is *not* true when n is unrestricted. These values of h are of course the numerically least positive and negative values of h that cause one of the derivatives to become infinite when we put $x = x_0 + h$.

Equation (9) is true even when h is taken *equal to the limiting value;* for, since the hypothesis of Art. 153 only requires that the derivatives shall not become infinite for values of x between x_0 and $x_0 + h$, this case is not excluded. [See examples 8 and 9, below.] If, however, it is the function itself (not merely one of its derivatives) that becomes infinite, equation (9) is inapplicable because R_0 is likewise infinite. [See Art. 154.]

When h is taken greater than the limiting value, equation (9) still holds true when $n + 1$ is less than the index of the first derivative that becomes infinite. [See example 10, below.]

158. In many cases the limit of convergence is numerically less than the limiting value of h mentioned in the preceding article, which refers only to the applicability of the expression for R_0; since, for certain values of h, although the expression for R_0 is applicable, *its value may fail to decrease as* n *increases.* The existence of this case is generally indicated by an increase in the value of the successive terms of the series. See Art. 160. Example 7, below, illustrates the applicability of equation (9) to a case in which h exceeds the limit of convergence.

The Binomial Theorem.

159. We shall now apply Taylor's Theorem to the function $(a + b)^m$ in order to obtain a series involving ascending powers of b.

In this case b takes the place of h, and a that of x_0; hence

$$f(x)=x^m \qquad \therefore\ f(x_0)=x_0^m \qquad =a^m$$

$$f'(x)=mx^{m-1} \qquad \therefore\ f'(x_0)=mx_0^{m-1} \qquad =ma^{m-1}$$

$$f''(x)=m(m-1)x^{m-2} \quad \therefore\ f''(x_0)=m(m-1)x_0^{m-2}=m(m-1)a^{m-2}$$

and

$$f^n(x_0)=m(m-1)(m-2)\cdots(m-n+1)a^{m-n}.$$

Whence

$$(a+b)^m=a^m+ma^{m-1}b+\frac{m(m-1)}{1\cdot 2}a^{m-2}b^2\ \cdots\cdot$$
$$+\frac{m(m-1)(m-2)\cdots(m-n+1)}{1\cdot 2\cdot 3\cdots n}a^{m-n}b^n+\cdots$$

This result is called the *Binomial Theorem.*

Each term of the series can be derived from the preceding term by multiplying by a factor of the form

$$\frac{m-n+1}{n}\cdot\frac{b}{a}=\left[\frac{m+1}{n}-1\right]\frac{b}{a}.$$

If m is a positive integer, this factor will become zero for $n=m+1$. When this is the case, the series will consist of a finite number of terms; otherwise, as n increases, the quantity in brackets will approach indefinitely to -1, and the terms will ultimately decrease numerically if b is less than a, but they will ultimately increase numerically if b exceeds a.

160. Assuming a positive, if b is likewise positive, the hypothesis of Art. 153 holds for all values of n; but it does not hold when b is negative and numerically greater than a, since $b=-a$ makes $f^n(a+b)=\infty$ when $n>m$. Hence for all values of b algebraically greater than $-a$ we are entitled

to use Lagrange's expression for the remainder, which gives in this case

$$R_0 = \frac{m(m-1)(m-2)\cdots(m-n)}{1\cdot 2\cdots(n+1)}(a+\theta b)^{m-n-1}b^{n+1}.$$

The ratio of the remainder to the first term omitted is

$$\frac{(a+\theta b)^{m-n-1}}{a^{m-n-1}} = \left(\frac{a}{a+\theta b}\right)^{n+1-m}.$$

The value of the fraction $\dfrac{a}{a+\theta b}$ is always between unity and $\dfrac{a}{a+b}$; hence, when the terms of the series decrease in value indefinitely, so likewise will the successive values of the remainder; that is, the series will be convergent. Therefore the limit of convergence for positive values of b is $b = a$. See Art. 158.

Examples XXI.

1. To expand $\log(x_0 + h)$ by Taylor's Theorem.

Solution :—

$$f(x_0 + h) = \log(x_0 + h)$$

$$f(x) = \log x \quad \therefore \quad f(x_0) = \log x_0$$

$$f'(x) = \frac{1}{x} \quad \therefore \quad f'(x_0) = \frac{1}{x_0}$$

$$f''(x) = -\frac{1}{x^2} \quad \therefore \quad f''(x_0) = -\frac{1}{x_0^2}$$

$$f'''(x) = \frac{2}{x^3} \quad \therefore \quad f'''(x_0) = \frac{1\cdot 2}{x_0^3}$$

$$f^{\text{iv}}(x) = -\frac{2\cdot 3}{x^4} \quad \therefore \quad f^{\text{iv}}(x_0) = -\frac{1\cdot 2\cdot 3}{x_0^4}$$

. . . .

$$f^n(x) = -(-1)^n \frac{1\cdot 2 \cdots (n-1)}{x^n} \therefore f^n(x_0) = -(-1)^n \frac{1\cdot 2 \cdots (n-1)}{x_0^n}.$$

By substituting in equation (4), Art. 152, we obtain

$$\log(x_0 + h) = \log x_0 + \frac{h}{x_0} - \frac{h^2}{2x_0^2} + \frac{h^3}{3x_0^3} - \frac{h^4}{4x_0^4} \cdots -(-1)^n \frac{h^n}{nx_0^n} + R_0.$$

Employing Lagrange's expression for the remainder (Art. 155) we derive

$$R_0 = (-1)^n \frac{h^{n+1}}{(n+1)(x_0 + \theta h)^{n+1}}.$$

2. Expand a^{x_0+h}.

Solution:—

$$f(x_0 + h) = a^{x_0+h}$$

$$f(x) = a^x \qquad \therefore \qquad f(x_0) = a^{x_0}$$

$$f'(x) = \log a \cdot a^x \qquad \therefore \qquad f'(x_0) = \log a \cdot a^{x_0}$$

$$f''(x) = (\log a)^2 \cdot a^x \qquad \therefore \qquad f''(x_0) = (\log a)^2 \cdot a^{x_0}$$

. . . .

$$f^n(x) = (\log a)^n \cdot a^x \qquad \therefore \qquad f^n(x_0) = (\log a)^n \cdot a^{x_0}.$$

Substituting in equation (4), Art. 152, we have

$$a^{x_0+h} = a^{x_0}\left[1 + \log a \cdot h + (\log a)^2 \frac{h^2}{1\cdot 2} \cdots + \frac{(\log a)^n \cdot h^n}{1\cdot 2 \cdots n}\right] + R_0.$$

Equation (10), Art. 156, gives

$$R_0 = \theta (\log a)^n a^{x_0} [a^h - 1] \frac{h^n}{1\cdot 2 \cdots n}.$$

3. Find the expansion of $f(x_0 + h)$, when $f(x) = x \log x - x$, writing the $(n+1)^{\text{th}}$ term of the series.

$$f(x_0 + h) = x_0 \log x_0 - x_0 + \log x_0 \cdot h + \frac{1}{x_0} \cdot \frac{h^2}{1 \cdot 2} - \frac{1}{x_0^2} \cdot \frac{h^3}{2 \cdot 3} + \cdots$$

$$+ (-1)^n \frac{1}{x_0^{n-1}} \cdot \frac{h^n}{(n-1)n} \cdots$$

4. Expand $\sin^{-1}(x_0 + h)$ to the fourth term inclusive.

$$\sin^{-1}(x_0 + h) = \sin^{-1} x_0 + \frac{h}{(1 - x_0^2)^{\frac{1}{2}}} + \frac{x_0}{(1 - x_0^2)^{\frac{3}{2}}} \cdot \frac{h^2}{1 \cdot 2}$$

$$+ \frac{1 + 2x_0^2}{(1 - x_0^2)^{\frac{5}{2}}} \cdot \frac{h^3}{1 \cdot 2 \cdot 3} + \cdots$$

5. Prove that

$$\sin\left(\tfrac{1}{6}\pi + h\right) = \tfrac{1}{2}\left[1 + h\sqrt{3} - \frac{h^2}{1 \cdot 2} - \frac{h^3\sqrt{3}}{1 \cdot 2 \cdot 3} + \frac{h^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{h^5\sqrt{3}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \cdots\right]$$

6. Prove that $f = \tan x,\ f' = \frac{1}{\cos^2 x},\ f'' = \frac{2 \sin x}{\cos^3 x},\ f''' = \frac{6}{\cos^4 x} - \frac{4}{\cos^2 x},\ f^{IV} = 8 \sin x \left(\frac{3}{\cos^5 x} - \frac{1}{\cos^3 x}\right)$

$$\tan\left(\tfrac{1}{4}\pi + h\right) = 1 + 2h + 2h^2 + \tfrac{8}{3}h^3 + \tfrac{10}{3}h^4 + \cdots$$

7. Find the value of the proper fraction θ which satisfies equation (9), Art. 155, when $f(x) = x^{\frac{5}{2}}$, assuming $x_0 = 1$, $h = 3$, and $n = 2$; also find θ for the same values of x_0 and h, when $n = 3$.

For $n = 2$, $\theta = \dfrac{6989}{33708}$; for $n = 3$, $\theta = 0{\cdot}15$ nearly.

8. Expand $(-1 + h)^{\frac{7}{3}}$, and show that the limit of applicability of equation (9), Art. 155, [see Art. 157] is $h = 1$: find also, for this value of h, the value of θ when $n = 2$, and when $n = 3$.

For $n = 2$, $\theta = 1 - \frac{2}{27}\sqrt{7}$; for $n = 3$, $\theta = 1 - \left(\frac{7}{18}\right)^{\frac{3}{4}}$.

9. Given $f(x) = x \log x - x$, show that the limit of applicability of equation (9), Art. 155, to $f(1 + h)$, for negative values of h, is $h = -1$,

and determine the values of θ for this value of h, when $n = 1$, and when $n = 2$. (See example 3.)

For $n = 1$, $\theta = \frac{1}{2}$; for $n = 2$, $\theta = 1 - \frac{1}{3}\sqrt{3}$.

10. In the case of the function $(-1 + h)^{\frac{7}{3}}$, show that if $h = 2$, equation (9), Art. 155, is not necessarily true when n exceeds unity, and determine the value of θ corresponding to $n = 1$. $\theta = \frac{158}{343}$.

11. By putting $x_0 = 1$ and $h = 1$ in the series obtained in example 3, derive a numerical series for computing $2 \log 2$, and, making $n = 6$, show by the method of Art. 156 that $2 \log 2$ is between 1.400 and 1.368.

XXII.

Maclaurin's Theorem.

161. We shall now give a particular form of Taylor's Series, which is usually more convenient, when numerical results are to be obtained, than the general form given in the preceding section.

This form of the series is obtained by putting $x_0 = 0$ and replacing h by x in equation (4), Art. 152. Thus,

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{1 \cdot 2} \cdots + f^n(0)\frac{x^n}{1 \cdot 2 \cdots n} + R_0 \quad . \; . \; (1)$$

and, the same substitutions being made in equation (8), Art. 155, we obtain

$$R_0 = f^{n+1}(\theta x)\frac{x^{n+1}}{1 \cdot 2 \cdots (n+1)}.$$

In like manner, by means of the result obtained in Art. 156, it may be shown that the value of $f(x)$ is between that of the two expressions,

$$f(0) + f'(0)x \cdots + f^n(0)\frac{x^n}{1\cdot 2\cdots n},$$

and
$$f(0) + f'(0)x \cdots + f^n(x)\frac{x^n}{1\cdot 2\cdots n}.$$

When $f(x)$ is denoted by y, equation (1) is written thus:—

$$y = y_0 + \frac{dy}{dx}\bigg]_0 x + \frac{d^2y}{dx^2}\bigg]_0 \frac{x^2}{1\cdot 2} + \cdots \quad . \quad . \quad . \quad . \quad (2)$$

162. The series given in equations (1) and (2), although first discovered by Stirling, has received the name of Maclaurin's Series. It was published by Maclaurin in 1742; but it does not appear that he ever intended to claim it as his own discovery.

As a mode of development Maclaurin's Theorem is in reality no less general than that of Taylor; for any function which is included in the general form $f(x_0 + h)$ may also, by giving a different signification to the symbol f, be expressed in the form $f(h)$, or, employing the notation of the last article, $f(x)$. Thus, if $\log(1 + h)$ is to be developed by Taylor's Theorem, $f(x) = \log x$, the value of x_0 being unity; but, if $\log(1 + x)$ is to be developed by Maclaurin's Theorem, we must put $f(x) = \log(1 + x)$. (Compare Ex. XXI., 1, with Art. 164.)

The Exponential Series and the Value of ε.

163. As an example of the application of the above theorem, we shall deduce the development of the function ε^x, which is called the *exponential series*, and shall thence obtain a series for computing the value of ε.

The successive derivatives of ε^x being equal to the original function, the coefficients, $f(0)$, $f'(0)$, etc., each reduce to unity;

we therefore derive, by substituting in equation (1) and introducing the value of R_0,

$$\varepsilon^x = 1 + x + \frac{x^2}{1\cdot 2} + \frac{x^3}{1\cdot 2\cdot 3} \cdots + \frac{x^n}{1\cdot 2\cdot \cdots \cdot n} + \varepsilon^{\theta x} \cdot \frac{x^{n+1}}{1\cdot 2\cdot \cdots (n+1)}.$$

Putting x equal to unity, we obtain the following series, which enables us to compute the value of the incommensurable quantity ε to any required degree of accuracy:

$$\varepsilon = 1 + 1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} \cdots$$
$$+ \frac{1}{1\cdot 2\cdot 3 \cdots n} + \frac{\varepsilon^\theta}{1\cdot 2\cdot 3 \cdots (n+1)}.$$

The computation may be arranged thus, each term being derived from the preceding term by division:

$$\begin{array}{r}
2.5 \\
.16666666667 \\
4166666667 \\
833333333 \\
138888889 \\
19841270 \\
2480159 \\
275573 \\
27557 \\
2505 \\
209 \\
16 \\
1 \\
\hline
2.71828182846
\end{array}$$

Since ε^θ is less than ε, the remainder (n being 14) is less than $\frac{3}{15}$ of the last term employed in the computation, and therefore cannot affect the result. Inasmuch as each term may contain a positive or negative error of one-half a unit in the last decimal

place, we cannot, in general, rely upon the accuracy of the last two places of decimals, in computations involving so large a number of terms. Accordingly, this computation only justifies us in writing

$$e = 2.718281828.$$

Logarithmic Series.

164. The logarithmic series is deduced by applying Maclaurin's Theorem to the function $\log(1 + x)$.

In this case

$$f(x) = \log(1 + x) \quad \therefore \quad f(0) = 0$$

$$f'(x) = \frac{1}{1 + x} \qquad \therefore \quad f'(0) = 1$$

$$f''(x) = -\frac{1}{(1 + x)^2} \quad \therefore \quad f''(0) = -1$$

$$f'''(x) = \frac{1 \cdot 2}{(1 + x)^3} \qquad \therefore \quad f'''(0) = 1 \cdot 2$$

$$f^{\text{iv}}(x) = -\frac{1 \cdot 2 \cdot 3}{(1 + x)^4} \quad \therefore \quad f^{\text{iv}}(0) = -1 \cdot 2 \cdot 3,$$

hence $$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad . \quad . \quad . \quad (1)$$

Since this series is divergent for values of x greater than unity (see Art. 158), we proceed to deduce a formula for the difference of two logarithms, which may be employed in computing successive logarithms; that is, denoting the numbers corresponding to two logarithms by n and $n + h$, we derive a series for

$$\log(n + h) - \log n = \log\frac{n + h}{n}.$$

A series which could be employed for this purpose might be obtained from (1), by putting $\frac{n+h}{n}$ in the form $1+\frac{h}{n}$. We obtain, however, a much more rapidly converging series by the process given below.

Substituting $-x$ for x in (1), we have

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots \quad . \quad . \quad (2)$$

Subtracting (2) from (1),

$$\log\frac{1+x}{1-x} = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\right], \quad . \quad . \quad (3)$$

a series involving only the positive terms of series (1).

Putting $\frac{1+x}{1-x} = \frac{n+h}{n}$, we derive $x = \frac{h}{2n+h}$; substituting in (3), we have

$$\log\frac{n+h}{n} = 2\left[\frac{h}{2n+h} + \frac{1}{3}\frac{h^3}{(2n+h)^3} + \frac{1}{5}\frac{h^5}{(2n+h)^5} + \cdots\right]. \quad . \quad (4)$$

The Computation of Napierian Logarithms.

165. The series given above enables us to compute Napierian logarithms. We proceed to illustrate by computing $\log_e 10$. The approximate numerical value of this logarithm could be obtained by putting $n=1$ and $h=9$ in (4); but, since the series thus obtained would converge very slowly, it is more convenient first to compute log 2 by means of the series obtained by putting $n=1$ and $h=1$ in (4); thus:

$$\log_e 2 = 2\left[\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3^3} + \frac{1}{5}\cdot\frac{1}{3^5} + \frac{1}{7}\cdot\frac{1}{3^7} + \cdots\right].$$

We then put $n = 8$ and $h = 2$ in (4); whence

$$\log_e 10 = 3\ \log_e 2 + \frac{2}{3}\left[\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3^6} + \frac{1}{5}\cdot\frac{1}{3^9} + \frac{1}{7}\cdot\frac{1}{3^{12}} + \cdots\right].$$

In making the computation, it is convenient first to obtain the values of the powers of $\frac{1}{3}$ which occur in the series for log 2, by successive division by 9, and afterwards to derive the values of the required terms of the series by dividing these auxiliary numbers by 1, 3, 5, 7, etc. The same auxiliary numbers are also used in the computation of $\log_e 10$. See the arrangement of the numerical work below.

$\frac{1}{3}$	0.3333333333	:	1	0.3333333333
$(\frac{1}{3})^3$	370370370	:	3	123456790
$(\frac{1}{3})^5$	41152263	:	5	8230453
$(\frac{1}{3})^7$	4572474	:	7	653211
$(\frac{1}{3})^9$	508053	:	9	56450
$(\frac{1}{3})^{11}$	56450	:	11	5132
$(\frac{1}{3})^{13}$	6272	:	13	482
$(\frac{1}{3})^{15}$	697	:	15	46
$(\frac{1}{3})^{17}$	77	:	17	5
				0.3465735902
				2
			$\log_e 2 =$	0.6931471804

$\frac{1}{3}$	0.3333333333	:	1	0.3333333333
$(\frac{1}{3})^5$	41152263	:	3	13717421
$(\frac{1}{3})^9$	508053	:	5	101611
$(\frac{1}{3})^{13}$	6272	:	7	896
$(\frac{1}{3})^{17}$	77	:	9	9
				0.3347153270
				0.1115717757
				0.2231435513
			$3\ \log_e 2 =$	2.0794415412
			$\log_e 10 =$	2.30258509

166. The common or tabular logarithms, of which 10 is the base, are derived from the corresponding Napierian logarithms by means of the relation

$$\log_e x = \log_e 10 \log_{10} x,$$

whence $$\log_{10} x = \frac{1}{\log_e 10} \cdot \log_e x = M . \log_e x.$$

The constant $\dfrac{1}{\log_e 10}$, denoted above by M, is called the *modulus* of common logarithms. Taking the reciprocal of $\log_e 10$, computed above, we have

$$M = 0.43429448.$$

The Developments of the Sine and the Cosine.

167. Let $$f(x) = \sin x,$$

then

$$f'(x) = \cos x, f''(x) = -\sin x, f'''(x) = -\cos x, f^{\text{iv}}(x) = \sin x;$$

f^{iv} being identical with f, it follows that these functions recur in cycles of four; their values when $x = 0$ are

$$0, 1, 0, -1, \text{etc.}$$

Hence substituting in equation (1), Art. 161, we have

$$\sin x = x - \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^5}{1 \cdot 2 \cdots 5} - \frac{x^7}{1 \cdot 2 \cdots 7} + \cdots \quad . \quad . \quad (1)$$

In a similar manner, we obtain

$$\cos x = 1 - \frac{x^2}{1 \cdot 2} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{x^6}{1 \cdot 2 \cdots 6} + \cdots \quad . \quad . \quad (2)$$

Huygens' Approximation to the Length of a Circular Arc.

168. A convenient formula known as "Huygens' Approximation to the length of a circular arc" is derived by means of the series for $\sin x$.*

Let s denote the length of the required arc, C its chord, c the chord of half the arc, and a the radius of the circle. Then

$$\frac{C}{a} = 2\sin\frac{s}{2a} \quad \text{and} \quad \frac{c}{a} = 2\sin\frac{s}{4a}.$$

Developing by the series given in equation (1) of the preceding article, we obtain

$$\frac{C}{2a} = \frac{s}{2a} - \frac{s^3}{1\cdot2\cdot3\cdot2^3a^3} + \frac{s^5}{1\cdot2\cdots5\cdot2^5a^5}\cdots, \quad . \; . \quad (1)$$

and
$$\frac{c}{2a} = \frac{s}{4a} - \frac{s^3}{1\cdot2\cdot3\cdot2^6a^3} + \frac{s^5}{1\cdot2\cdots5\cdot2^{10}a^5}\cdots \quad . \; . \quad (2)$$

Multiplying equation (2) by 8, and subtracting equation (1) to eliminate the term containing s^3, we have, approximately,

$$\frac{8c - C}{2a} = \frac{3s}{2a} - \frac{3}{4}\cdot\frac{s^5}{1\cdot2\cdots5\cdot2^5a^5},$$

whence
$$\frac{8c - C}{3} = s - \frac{s^5}{7680\,a^4}, \quad . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

and, omitting the term containing s^5,

$$s = \frac{8c - C}{3} = 2c + \tfrac{1}{3}(2c - C).$$

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* This article is taken with slight modifications from *Williamson's Differential Calculus*, third edition, London, 1877.

The error committed in adopting this value for s is less than the term containing s^5 in equation (3), since the remainder neglected in writing this equation is positive.

Examples XXII.

1. Expand $(1 + x)^m$.

$$(1 + x)^m = 1 + mx + \frac{m(m-1)}{1\cdot 2}x^2 + \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}x^3 + \cdots$$

It is evident that no coefficient will vanish if m is negative or fractional. This is the form in which the binomial theorem is employed in computation, x being less than unity.

2. Find three terms of the expansion of $\sin^2 x$.

$$\sin^2 x = x^2 - \frac{x^4}{3} + \frac{2x^6}{3^2\cdot 5} - \cdots$$

3. Expand $\tan x$ to the term involving x^5 inclusive.

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$

4. Expand $\sec x$ to the term involving x^6 inclusive.

$$\sec x = 1 + \frac{x^2}{1\cdot 2} + \frac{5x^4}{1\cdot 2\cdot 3\cdot 4} + \frac{61x^6}{1\cdot 2\cdot\cdots\cdot 6} + \cdots$$

5. Expand $\log\sec x$ to the term involving x^6 inclusive.

$$\log\sec x = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \cdots$$

6. Find four terms of the expansion of $\varepsilon^x \sec x$.

$$\varepsilon^x \sec x = 1 + x + x^2 + \frac{2x^3}{3} + \cdots$$

7. Derive the expansion of $\log(1 - x^2)$ from the logarithmic series, and verify by adding the expansions of $\log(1 + x)$ and $\log(1 - x)$.

8. Expand $\log(1 - x + x^2)$.

We may put $1 - x + x^2$ *in the form* $\dfrac{1 + x^3}{1 + x}$, *and employ the logarithmic series.*

$$\log(1 - x + x^2) = -x + \frac{x^2}{2} + \frac{2x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5} - \frac{x^6}{3} - \cdots$$

9. Derive the expansion of $(1 + x)\varepsilon^x$ from that of ε^x.

$$(1 + x)\varepsilon^x = 1 + 2x + \frac{3 \cdot x^2}{1 \cdot 2} \cdots + \frac{n + 1}{1 \cdot 2 \cdots n} x^n.$$

10. Find, by means of the exponential series, the expansion of $x\varepsilon^{2x}$, including the nth term.

11. Expand $\dfrac{\varepsilon^x}{1 + x}$ by division, making use of the exponential series.

$$\frac{\varepsilon^x}{1 + x} = 1 + \frac{x^2}{2} - \frac{x^3}{3} + \frac{3x^4}{8} - \frac{11x^5}{30} + \cdots$$

12. Find the expansion of $\varepsilon^x \log(1 + x)$ to the term involving x^5, by multiplying together a sufficient number of the terms of the series for ε^x and for $\log(1 + x)$.

$$\varepsilon^x \log(1 + x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{3x^5}{40} + \cdots$$

13. Expand $\log(1 + \varepsilon^x)$.

$$\log(1 + \varepsilon^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \cdots$$

14. Expand $(1 + \varepsilon^x)^n$ to the term involving x^3 inclusive.

$$(1 + \varepsilon^x)^n = 2^n \left\{ 1 + \frac{n}{2} \cdot x + \frac{n(n + 1)}{2^2} \cdot \frac{x^2}{1 \cdot 2} + \frac{n(n^2 + n + 2)}{2^3} \cdot \frac{x^3}{1 \cdot 2 \cdot 3} + \cdots \right\}.$$

15. Find the expansion of $\sqrt{(1 \pm \sin 2x)}$, employing the formula $\sqrt{(1 \pm \sin 2x)} = \cos x \pm \sin x$.

$$\sqrt{(1 \pm \sin 2x)} = 1 \pm x - \frac{x^2}{1 \cdot 2} \mp \frac{x^3}{1 \cdot 2 \cdot 3} + \cdots$$

16. Find the expansion of $\cos^2 x$ by means of the formula $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$.

$$\cos^2 x = 1 - x^2 + \frac{2^3 x^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{2^5 x^6}{1 \cdot 2 \cdots 6} + \cdots$$

17. Find the expansion of $\cos^3 x$, by means of the formula $\cos^3 x = \frac{1}{4}(\cos 3x + 3 \cos x)$.

$$\cos^3 x = 1 - \frac{3x^2}{1 \cdot 2} + \frac{3^4 + 3}{4} \cdot \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} \cdots + (-1)^n \frac{3^{2n} + 3}{4} \cdot \frac{x^{2n}}{1 \cdot 2 \cdots 2n}.$$

18. Compute $\log_e 3$, and find $\log_{10} 3$ by multiplying by the value of M (Art. 166).

$\log_e 3 = 1.0986123.$
$\log_{10} 3 = 0.4771213.$

19. Find $\log_e 269$.

Put $n = 270 = 10 \times 3^3$, *and* $h = -1$.

$\log_e 269 = 5.5947114.$

20. Find $\log_e 7$, and $\log_e 13$.

$\log_e 7 = 1.9459101.$
$\log_e 13 = 2.5649494.$

21. The chord of an arc, supposed to be circular, is found by measurement to be 5176.4 ft., and the chord of half the arc to be 2610.5 ft.; find the length of the arc by Huygens' approximation.

$s = 5235.9$ ft.

XXIII.

Even Functions and Odd Functions.

169. Many simple functions have the property expressed by the equation

$$f(-x) = f(x), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and are called *even functions:* for example, $\cos x$, $\log(1 + x^2)$, and $a + x^2 + bx^4$, are even functions.

Every rational integral function involving powers of x with even exponents only evidently has this property; and it is also evident that the series corresponding to any function having this property can consist only of such terms.

Certain other functions have the property expressed by the equation

$$f(-x) = -f(x), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and are called *odd functions:* for example, $\sin x$, $\tan x$, and $ax + bx^3$ are odd functions. It is evident that the series corresponding to a function having the latter property can contain only odd powers of x.

These two classes do not include all functions; in fact, it is evident that many functions satisfy neither (1) nor (2): the developments of such functions contain of course both odd and even powers.

170. In the case of an odd function, which does not become infinite for $x = 0$, we have, from (2),

$$f(0) = -f(0) \quad \therefore \quad 2f(0) = 0;$$

therefore, when $f(x)$ denotes an odd function,

$$f(0) = 0 \qquad \text{or} \qquad f(0) = \infty.$$

The derivative of an even function is an odd function; for, if we take the derivative of

$$f(x) = f(-x),$$

we obtain
$$f'(x) = -f'(-x).$$

In like manner, it may be proved that *the derivative of an odd function is an even function.*

It follows that in the case of an *even function* all the derivatives of an *odd* order vanish or become infinite for $x = 0$, and that in the case of an *odd function* the *even* derivatives vanish or become infinite.

The Development of the Inverse Tangent.

171. When the series of functions f, f', f'', etc., does not follow an obvious law of formation, the labor of deriving the terms of the development of $f(x)$ by the ordinary process increases rapidly with the number of terms required. If however the development of $f'(x)$ is known that for $f(x)$ may be found in the manner illustrated below.

Let
$$f(x) = \tan^{-1} x,$$

then
$$f'(x) = \frac{1}{1 + x^2},$$

which may be developed by division. Since the primary value of $\tan^{-1}x$ is an odd function, we assume for this value

$$\tan^{-1} x = Ax + Bx^3 + Cx^5 + Dx^7 + \cdots; \quad . \quad . \quad . \quad (1)$$

by taking derivatives, we have

$$\frac{1}{1 + x^2} = A + 3Bx^2 + 5Cx^4 + 7Dx^6 + \cdots;$$

but by division

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

These developments must be identical; hence, we have

$$A = 1, \quad B = -\tfrac{1}{3}, \quad C = \tfrac{1}{5}, \quad \text{etc.}$$

and therefore, substituting in (1),

$$\tan^{-1} x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \cdots$$

When the series for $f(x)$ is known, the series for $f'(x)$ may be at once derived from it by differentiation. Thus the series for $\cos x$, Art. 167, may be obtained by differentiating the series for $\sin x$.

The Development of Implicit Functions by Maclaurin's Theorem.

172. The process of evaluating the derivatives in the case of implicit functions is exemplified below. A few terms only of the development are usually found in this way, as the process is necessarily tedious.

Let $$y^3 - 6xy - 8 = 0; \quad \ldots\ldots \quad (1)$$

whence, differentiating,

$$(y^2 - 2x)\frac{dy}{dx} - 2y = 0, \quad \ldots\ldots \quad (2)$$

and $$2y\left(\frac{dy}{dx}\right)^2 - 4\frac{dy}{dx} + (y^2 - 2x)\frac{d^2y}{dx^2} = 0. \quad \ldots\ldots \quad (3)$$

Putting $x = 0$ in equation (1), we obtain

$$y_0 = 2 \text{ (the only real value of } y_0\text{)},$$

and, substituting this value for y in equation (2), we obtain

$$\left.\frac{dy}{dx}\right]_0 = 1,$$

and in like manner, by substituting in each of the successive differential equations the values of the derivatives already found, we obtain

$$\left.\frac{d^2y}{dx^2}\right]_0 = 0, \qquad \left.\frac{d^3y}{dx^3}\right]_0 = -\tfrac{1}{2}, \qquad \left.\frac{d^4y}{dx^4}\right]_0 = 1, \text{ etc.}$$

Whence, substituting in equation (2), Art. 161, we have

$$y = 2 + x - \tfrac{1}{2}\cdot\frac{x^3}{1\cdot 2\cdot 3} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4}\cdots$$

Evaluation by Development in Series.

173. When a function $\dfrac{f(x)}{\phi(x)}$ takes the form $\dfrac{0}{0}$ for $x = 0$, its evaluation is frequently facilitated by developing the terms $f(x)$ and $\phi(x)$ in series involving powers of x.

Let the result of the development be

$$\frac{f(x)}{\phi(x)} = \frac{A_0 + A_1 x + A_2 x^2 + \cdots}{a_0 + a_1 x + a_2 x^2 + \cdots}.$$

Since, by hypothesis $f(0) = 0$ and $\phi(0) = 0$, $A_0 = 0$ and $a_0 = 0$; we may also find that one or more of the coefficients A_1, A_2, etc., and a_1, a_2, etc. vanish. Let x^m and x^n be the lowest powers of x which remain in the numerator and in the denominator respectively; then,

$$\frac{f(x)}{\phi(x)} = \frac{x^m}{x^n}\cdot\frac{A_m + A_{m+1}x + \cdots}{a_n + a_{n+1}x + \cdots}.$$

When $x = 0$, the second factor has the finite value $\dfrac{A_m}{a_n}$; hence

$$\left.\frac{f(x)}{\phi(x)}\right]_0 = \frac{A_m}{a_n} \cdot \left.\frac{x^m}{x^n}\right]_0.$$

The value of this ratio is finite when its terms are of the same degree, but is either 0 or ∞ when these terms are not of the same degree.

174. When this method is employed, we are sometimes able to determine at the outset the highest power of x which it is necessary to retain, as in the following example.

Let it be required to determine the value of the function

$$\left.\frac{x \sin(\sin x) - \sin^2 x}{x^6}\right]_0.$$

It is unnecessary in this case to retain in the development of the numerator any term whose degree is higher than the sixth; and hence, in that of $\sin(\sin x)$, no terms need be retained higher in degree than x^5. Employing the series for $\sin x$, Art. 167, we have

$$\sin(\sin x) = \sin x - \tfrac{1}{6}\sin^3 x + \tfrac{1}{120}\sin^5 x - \cdots,$$

$$= (x - \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5 \cdots) - \tfrac{1}{6}(x^3 - \tfrac{1}{2}x^5 \cdots) + \tfrac{1}{120}x^5 \cdots$$

$$= x - \tfrac{1}{3}x^3 + \tfrac{1}{10}x^5 \cdots;$$

and $$x \sin(\sin x) = x^2 - \tfrac{1}{3}x^4 + \tfrac{1}{10}x^6 \cdots;$$

also, squaring the series for $\sin x$,

$$\sin^2 x = x^2 - \tfrac{1}{3}x^4 + \tfrac{2}{45}x^6 \cdots;$$

hence $$x \sin(\sin x) - \sin^2 x = \tfrac{1}{18}x^6 \cdots$$

The value of the given fraction is therefore $\frac{1}{18}$.

Had we found the development of the numerator to contain terms lower in degree than x^6, the value of the fraction would have been infinite; but had the term in x^6 vanished, the value of the fraction would have been zero.

175. The method of development may also be used to furnish a demonstration of the ordinary formula for the evaluation of a function which takes the indeterminate form.

By Taylor's theorem, using Lagrange's form of the remainder, we have

$$\frac{f(a+h)}{\phi(a+h)} = \frac{f(a) + f'(a+\theta h)\cdot h}{\phi(a) + \phi'(a+\theta h)\cdot h};$$

when $f(a) = 0$ and $\phi(a) = 0$, this equation reduces to

$$\frac{f(a+h)}{\phi(a+h)} = \frac{f'(a+\theta h)}{\phi'(a+\theta h)};$$

hence, putting $h = 0$,

$$\frac{f(a)}{\phi(a)} = \frac{f'(a)}{\phi'(a)},$$

a result identical with that obtained by a different process in Art. 95.

Symbolic Form of Taylor's Theorem.

176. By employing the notation $\frac{d}{dx}$, $\frac{d^2}{dx^2}$, etc., for the derivatives, we may write Taylor's theorem in the form

$$f(x+h) = f(x) + h\frac{d}{dx}f(x) + \frac{h^2}{1\cdot 2}\cdot\frac{d^2}{dx^2}f(x) + \frac{h^3}{1\cdot 2\cdot 3}\cdot\frac{d^3}{dx^3}f(x) + \cdots$$

The form

$$f(x+h) = \left[1 + h\frac{d}{dx} + \frac{h^2}{1\cdot 2}\cdot\frac{d^2}{dx^2} + \frac{h^3}{1\cdot 2\cdot 3}\cdot\frac{d^3}{dx^3} + \ldots\right]f(x),$$

in which $f(x)$ is removed from within the brackets as if it were a factor, may also be employed, inasmuch as this form of writing the second member admits of no ambiguity of meaning. It will be noticed that the terms within the brackets proceed according to the law of the exponential series (Art. 163); we may therefore write

$$f(x+h) = \epsilon^{h\frac{d}{dx}} f(x),$$

which is to be understood as implying that $\epsilon^{h\frac{d}{dx}}$ is to be developed in accordance with the form of the exponential series, and $f(x)$ introduced after each term.

Examples XXIII.

1. Prove that each of the following expressions is an *even* function of x; viz.,—

$$\sin x^2, \quad (\sin^{-1}x)^2, \quad \epsilon^{-\frac{1}{x^2}}.$$

2. Prove that each of the following expressions is an *odd* function of x; viz.,—

$$\log\frac{1+x}{1-x}, \qquad \text{and} \qquad \frac{\epsilon^x+1}{\epsilon^x-1}.$$

3. Prove that the following functions are even; viz.,—

$$x\cot x, \qquad \text{and} \qquad \frac{x}{2}+\frac{x}{\epsilon^x-1}.$$

4. Prove that the following functions are odd; viz.,—

$$\log\tan(\tfrac{1}{4}\pi+x), \qquad \text{and} \qquad \log[\sqrt{(1+x^2)}+x].$$

5. Prove that, when $\phi(x)$ is an even function, $f[\phi(x)]$ is likewise an even function, f denoting any function.

6. Prove that, if $f(x)$ and $\phi(x)$ are both odd functions, $f[\phi(x)]$ is an odd function.

7. Deduce the expansion of $\log(1 + x)$ by the method employed in deducing the expansion of $\tan^{-1} x$ in Art. 171.

8. Deduce the expansion of ε^x by means of an assumed series.

Compare the series for $f'(x)$ with the series assumed for $f(x)$.

9. Expand $\sin^{-1} x$ by the method of Art. 171.

The expansion of $f'(x)$ may be effected by the binomial theorem.

$$\sin^{-1}x = x + \tfrac{1}{2}\cdot\tfrac{1}{3}x^3 + \tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{1}{5}x^5 + \tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{5}{6}\cdot\tfrac{1}{7}x^7 + \cdots$$

10. Derive by differentiation three terms of the expansion of $\tan x$ from that of $\log\sec x$ [Ex. XXII, 5], and from the result derive the expansion of $\sec^2 x$.

11. Expand $\varepsilon^{\varepsilon^x}$ to the term containing x^3 inclusive.

We easily deduce $f'(x) = \varepsilon^x f(x)$; therefore, the product of the assumed series and the exponential series must be equivalent to the derivative of the former.

$$\varepsilon^{\varepsilon^x} = \varepsilon[1 + x + x^2 + \tfrac{5}{6}x^3 + \cdots]$$

12. Given $\varepsilon^y + xy = \varepsilon$, to find the expansion of y in powers of x.

$$y = 1 - \frac{x}{\varepsilon} + \frac{1}{1\cdot 2}\cdot\frac{x^2}{\varepsilon^2} + \frac{1}{1\cdot 2\cdot 3}\cdot\frac{x^3}{\varepsilon^3} + \cdots$$

13. Given $y^2 - xy = 1$, to find the expansion of y in powers of x.

$$y = \pm 1 + \tfrac{1}{2}x \pm \frac{1}{2^2}\cdot\frac{x^2}{1\cdot 2} \mp \frac{3}{2^4}\cdot\frac{x^4}{1\cdot 2\cdot 3\cdot 4} \pm \cdots$$

14. Given $y = 1 + x\varepsilon^y$, to find the expansion of y in powers of x.

$$y = 1 + \varepsilon x + 2\varepsilon^2\cdot\frac{x^2}{1\cdot 2} + 9\varepsilon^3\cdot\frac{x^3}{1\cdot 2\cdot 3} + \cdots$$

Evaluate the following functions by the method of Art. 173.

15. $\dfrac{m\sin\theta - \sin m\theta}{\theta(\cos\theta - \cos m\theta)}$, when $\theta = 0$. $\tfrac{1}{3}m$.

16. $\dfrac{1}{x} - \dfrac{(1 + x)\log(1 + x)}{x^2}$, when $x = 0$. $-\frac{1}{2}$.

17. $\dfrac{(x + \sin 2x - 6 \sin \frac{1}{2}x)^2}{(4 + \cos x - 5 \cos \frac{1}{2}x)^3}$, $x = 0$. $\dfrac{8.29^2}{3^2}$.

18. $\dfrac{\tan \pi x - \pi x}{2x^2 \tan \pi x}$, $x = 0$. $\dfrac{\pi^2}{6}$.

19. $\dfrac{\theta\,(2\theta + \sin 2\theta - 4 \sin \theta)}{3 + \cos 2\theta - 4 \cos \theta}$, $\theta = 0$. $-\frac{4}{3}$.

XXIV.

The Development of Functions by Means of Differential Equations.

177. When a general expression for the nth derivative of a function cannot be obtained, it is, nevertheless, frequently possible to derive the particular values which the derivatives take when $x = 0$, by the method explained in Articles 91 and 92. Thus, in the case of the function

$$f(x) = y = \sin(m \sin^{-1} x),$$

by substituting in Maclaurin's series the numerical values of the derivatives when $x = 0$, as derived in Art. 92, we have [since $f(0) = 0$]

$$\sin(m \sin^{-1} x) = mx + \frac{m(1 - m^2)}{1\cdot2\cdot3}x^3 + \frac{m(1 - m^2)(9 - m^2)}{1\cdot2\cdot3\cdot4\cdot5}x^5 \ldots$$

This expansion is equivalent to that of $\sin mz$ in powers of $\sin z$; for, if we put $z = \sin^{-1} x$, the equation becomes

$$\sin mz = m \sin z\left[1 - \frac{m^2 - 1}{1\cdot2\cdot3}\sin^2 z + \frac{(m^2 - 1)(m^2 - 9)}{1\cdot2\cdot3\cdot4\cdot5}\sin^4 z - \ldots\right]. \quad (1)$$

This series will consist of a finite number of terms when m is an *odd* integer.

In a similar manner, it may be proved that

$$\cos mz = 1 - \frac{m^2}{1\cdot 2}\sin^2 z + \frac{m^2(m^2-4)}{1\cdot 2\cdot 3\cdot 4}\sin^4 z - \cdots, \quad . \quad . \quad (2)$$

the number of terms being finite when m is an even integer.

178. The complete process in the case of the function $(\sin^{-1} x)^2$ is given below, $\sin^{-1} x$ here denoting the primary value of this function.

Let $$y = (\sin^{-1} x)^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

then $$\frac{dy}{dx} = \frac{2\sin^{-1} x}{\sqrt{(1-x^2)}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and $$\frac{d^2y}{dx^2} = 2\,\frac{1 + \dfrac{x\sin^{-1} x}{\sqrt{(1-x^2)}}}{1-x^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Combining (2) and (3), to eliminate the transcendental functions and radicals, we obtain the differential equation

$$a_{n+2} = \frac{n^2}{(n+2)(n+1)}\,a_n$$

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2. \qquad a_0 = 0,\ a_1 = 0,\ a_2 = \frac{2}{2}$$

$$a_3 = 0,\ a_4 = \frac{4^2}{6\cdot 5}$$

Taking, by means of Leibnitz' theorem, the nth derivative of each term (see Art. 88), we have

$$(1-x^2)\frac{d^{n+2}y}{dx^{n+2}} - 2nx\frac{d^{n+1}y}{dx^{n+1}} - n(n-1)\frac{d^ny}{dx^n} - x\frac{d^{n+1}y}{dx^{n+1}} - n\frac{d^ny}{dx^n} = 0,$$

whence putting $x = 0$, we derive

$$\frac{d^{n+2}y}{dx^{n+2}} = n^2\frac{d^ny}{dx^n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Since from equation (2) we have

$$\left.\frac{dy}{dx}\right]_0 = 0,$$

equation (4) shows that the odd derivatives all vanish for $x = 0$, and also enables us to determine the values of the even derivatives, the value of the second derivative being first obtained from equation (3), thus :—

$$\frac{d^2y}{dx^2} = 2, \qquad \frac{d^4y}{dx^4} = 2\cdot 2^2, \qquad \frac{d^6y}{dx^6} = 2\cdot 2^2\cdot 4^2, \text{ etc.}$$

Finally, substituting in equation (2), Art. 161, we obtain the expansion,

$$(\sin^{-1}x)^2 = 2\left[\frac{x^2}{1\cdot 2} + \frac{2^2x^4}{1\cdot 2\cdot 3\cdot 4} + \frac{2^2\cdot 4^2x^6}{1\cdot 2\cdots 6} + \frac{2^2\cdot 4^2\cdot 6^2x^8}{1\cdot 2\cdots 8}\cdots\right]$$

$$= 2\left[\frac{x^2}{2} + \frac{2}{3}\cdot\frac{x^4}{4} + \frac{2\cdot 4}{3\cdot 5}\cdot\frac{x^6}{6} + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\cdot\frac{x^8}{8}\cdots\right].$$

The Computation of π.

179. By differentiating the series for $(\sin^{-1}x)^2$, obtained in the preceding article, we derive a convenient series for the computation of π; viz.,—

$$\frac{\sin^{-1}x}{\sqrt{(1-x^2)}} = x + \frac{2}{3}x^3 + \frac{2\cdot 4}{3\cdot 5}x^5 + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}x^7 + \cdots$$

This equation gives an expression for the circular measure of

an angle in the form of a series involving powers of its sine; for, putting $\sin^{-1}x = \theta$, we have

$$\theta = \cos\theta \sin\theta\left[1 + \frac{2}{3}\sin^2\theta + \frac{2\cdot4}{3\cdot5}\sin^4\theta + \cdots\right].$$

The arc may also be expressed in terms of its tangent; thus, since $\sin\theta = \tan\theta\cos\theta$,

$$\theta = \frac{\tan\theta}{\sec^2\theta}\left[1 + \frac{2}{3}\frac{\tan^2\theta}{\sec^2\theta} + \frac{2\cdot4}{3\cdot5}\frac{\tan^4\theta}{\sec^4\theta} + \cdots\right],$$

or, denoting $\tan\theta$ by x,

$$\tan^{-1}x = \frac{x}{1+x^2}\left[1 + \frac{2}{3}\frac{x^2}{1+x^2} + \frac{2\cdot4}{3\cdot5}\left(\frac{x^2}{1+x^2}\right)^2 + \cdots\right]. \quad . \quad . \quad (1)$$

180. This series is convergent for all values of x. Since $\tan\frac{1}{4}\pi = 1$, we can obtain a series for $\frac{1}{4}\pi$ by making $x = 1$. We are however enabled to use much more convergent series by employing the formula,

$$\frac{\pi}{4} = 5\tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{3}{79}.^* \quad . \quad . \quad . \quad . \quad (2)$$

* The employment of this formula in the computation of π was suggested by Euler in a memoir on the subject in 1779.

This, as well as many similar formulas which have been used for the same purpose, may be readily verified by means of the expression for the tangent of the sum of two arcs. Denoting the tangents of these arcs by m and n, the tangent of the sum is

$$\frac{m+n}{1-mn};$$

whence, putting $m = \frac{1}{7}$, and $n = \frac{3}{79}$, we obtain

$$\tan^{-1}\tfrac{1}{7} + \tan^{-1}\tfrac{3}{79} = \tan^{-1}\tfrac{2}{11}.$$

Hence $\qquad 5\tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{3}{79} = 3\tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{2}{11},$

in like manner the second member may be reduced to

$$\tan^{-1}\tfrac{1}{7} + 2\tan^{-1}\tfrac{1}{3}, \quad \text{and finally to} \quad \tan^{-1}1 \quad \text{or} \quad \tfrac{1}{4}\pi.$$

For $x = \frac{1}{7}$, the fraction

$$\frac{x^2}{1+x^2} = \frac{1}{50} = \frac{2}{100},$$

and, for $x = \frac{3}{79}$,

$$\frac{x^2}{1+x^2} = \frac{9}{6250} = \frac{144}{100000}.$$

The small values of these fractions render the series for $\tan^{-1}\frac{1}{7}$ and for $\tan^{-1}\frac{3}{79}$ rapidly convergent, and the computation is facilitated by the fact that the denominators are powers of ten.

181. Substituting the equivalent series for the inverse tangents in equation (2) and multiplying by 4, we have

$$\pi = \frac{28}{10}\left[1 + \frac{2}{3}\cdot\frac{2}{100} + \frac{2}{3}\cdot\frac{4}{5}\left(\frac{2}{100}\right)^2\cdots\right] + \frac{30336}{100000}\left[1 + \frac{2}{3}\cdot\frac{144}{100000} + \frac{2}{3}\cdot\frac{4}{5}\left(\frac{144}{100000}\right)^2\cdots\right].$$

This series may be written in the following form, in which each of the letters α, β, γ, etc., denotes the value of the term preceding that in which it occurs.

$$\pi = \frac{14}{15}\left[3 + \frac{4}{100} + \frac{16}{10}\left(\frac{2}{100}\right)^2 + \frac{6}{7}\cdot\frac{2\alpha}{100} + \frac{8}{9}\cdot\frac{2\beta}{100} + \frac{10}{11}\cdot\frac{2\gamma}{100}\cdots\right]$$
$$+ \frac{10112}{100000}\left[3 + \frac{288}{100000} + \frac{4}{5}\cdot\frac{144\alpha}{100000} + \frac{6}{7}\cdot\frac{144\beta}{100000}\cdots\right].$$

The numerical work for ten places of decimals is given below. The method by which each term is derived from the preceding term is indicated by the latter form of writing the series: multiplication by the fractions $\frac{6}{7}$, $\frac{8}{9}$, etc., is effected by deducting $\frac{1}{7}$, $\frac{1}{9}$, etc., of the quantity to be multiplied.

$$
\begin{array}{lr}
 & 3.04 \\
\alpha = & 0.00064 \\
 & 128 \\
 & 182857 \\
\hline
\beta = & 1097143 \\
 & 21943 \\
 & 2438 \\
\hline
\gamma = & 19505 \\
 & 390 \\
 & 35 \\
\hline
\delta = & 355 \\
 & 7 \\
 & 1 \\
\hline
\varepsilon = & 6 \\
\hline
 & 3.04065117009 \\
 & .20271007801 \\
\hline
 & 2.83794109208
\end{array}
\qquad\qquad
\begin{array}{lr}
 & 3. \\
\alpha = & 0.00288 \\
0.00144\,\alpha = & 41472 \\
 & 82944 \\
\hline
\beta = & 331776 \\
0.00144\,\beta = & 478 \\
 & 68 \\
\hline
\gamma = & 410 \\
\hline
 & 3.002883321186 \\
\hline
 & 30028833219 \\
 & 300288332 \\
 & 30028833 \\
 & 6005767 \\
\hline
 & 0.30365156151 \\
 & 2.83794109208 \\
\hline
\pi = & 3.1415926536
\end{array}
$$

Examples XXIV.

1. By means of the series given in Art. 177, derive values of $\sin 5x$ and of $\sin 7x$.

$$\sin 5x = 5\sin x - 20\sin^3 x + 16\sin^5 x;$$
$$\sin 7x = 7\sin x - 56\sin^3 x + 112\sin^5 x - 64\sin^7 x.$$

2. By means of the series given in Art. 177, derive values of $\cos 4x$ and of $\cos 6x$.

$$\cos 4x = 1 - 8\sin^2 x + 8\sin^4 x;$$
$$\cos 6x = 1 - 18\sin^2 x + 48\sin^4 x - 32\sin^6 x.$$

3. By means of formula (2), Art. 177, prove that

$$\cos z = 1 - \tfrac{1}{2}\sin^2 z - \tfrac{1}{8}\sin^4 z - \tfrac{1}{16}\sin^6 z - \cdots$$

4. By means of formula (2), Art. 177, derive four terms of the expansion for $\cos \frac{1}{2}z$.

$$\cos \tfrac{1}{2}z = 1 - \tfrac{1}{8}\sin^2 z - \tfrac{5}{128}\sin^4 z - \tfrac{21}{1024}\sin^6 z - \cdots$$

5. Expand $\tan^{-1}\dfrac{x}{a}$ by means of the nth derivative of this function. (*See Art.* 87, *and compare Ex. XII*, 20.)

$$\tan^{-1}\frac{x}{a} = \frac{x}{a} - \frac{x^3}{3a^3} + \frac{x^5}{5a^5} - \cdots$$

6. Expand $\epsilon^x \cos x$, by means of the nth derivative. (*See Art.* 85.)

$$\epsilon^x \cos x = 1 + x - \frac{2x^3}{1\cdot 2\cdot 3} - \frac{2^2 x^4}{1\cdot 2\cdots 4} - \frac{2^2 x^5}{1\cdot 2\cdots 5}$$
$$+ \frac{2^3 x^7}{1\cdot 2\cdots 7} + \frac{2^4 x^8}{1\cdot 2\cdots 8} + \cdots$$

7. Expand $\epsilon^x \sin x$. (*See Ex. XII*, 6.)

$$\epsilon^x \sin x = x + x^2 + \frac{2x^3}{1\cdot 2\cdot 3} - \frac{2^2 x^5}{1\cdot 2\cdots 5} - \frac{2^3 x^6}{1\cdot 2\cdots 6} - \frac{2^3 x^7}{1\cdot 2\cdots 7}\cdots$$

8. Expand $\epsilon^{x\cos\alpha}\cos(x\sin\alpha)$. (*See Ex. XII*, 8.)

$$\epsilon^{x\cos\alpha}\cos(x\sin\alpha) = 1 + \cos\alpha\cdot x + \cos 2\alpha\cdot\frac{x^2}{1\cdot 2}\cdots$$

9. Write the expansion of $\sin^{-1}x$, employing the results found in Ex. XII, 19.

$$\sin^{-1}x = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^7}{7} + \cdots$$

10. Write the expansion of $\log[x + \sqrt{(a^2 + x^2)}]$, employing the results found in Ex. XII, 21.

$$\log[x + \sqrt{(a^2 + x^2)}] = \log a + \frac{x}{a} - \frac{1}{2}\cdot\frac{x^3}{3a^3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5a^5} - \cdots$$

11. Write the expansion of $[x + \sqrt{(a^2 + x^2)}]^m$, employing the results found in Ex. XII, 22.

$$[x + \sqrt{(a^2 + x^2)}]^m = a^m + ma^{m-1}x + \frac{m^2}{1\cdot 2}a^{m-2}x^2$$
$$+ \frac{m(m^2-1)}{1\cdot 2\cdot 3}a^{m-3}x^3 + \frac{m^2(m^2-4)}{1\cdot 2\cdot 3\cdot 4}a^{m-4}x^4 + \cdots$$

12. Expand by Taylor's theorem the function

$$a\cos[\log(1+h)] + b\sin[\log(1+h)],$$

employing the results obtained in Ex. XII, 24.

$$a\cos[\log(1+h)] + b\sin[\log(1+h)] = a + bh - \frac{a+b}{2}h^2$$
$$+ \frac{3a+b}{1\cdot 2\cdot 3}h^3 - 10a\frac{h^4}{1\cdot 2\cdots 4} + \cdots$$

13. Expand $\varepsilon^{m\sin^{-1}x}$ by the method of Art. 178.

$$\varepsilon^{m\sin^{-1}x} = 1 + mx + \frac{m^2x^2}{1\cdot 2} + \frac{m(m^2+1)}{1\cdot 2\cdot 3}x^3 + \frac{m^2(m^2+4)}{1\cdot 2\cdot 3\cdot 4}x^4 + \cdots$$

14. Given the differential equation

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0,$$

and $$y_0 = a,$$

to expand y in powers of x.

$$y = a\left[1 - x + \frac{x^2}{4} - \frac{x^3}{4\cdot 9} + \frac{x^4}{4\cdot 9\cdot 16} - \frac{x^5}{4\cdot 9\cdot 16\cdot 25} + \cdots\right].$$

XXV.

Functions of Imaginary Quantities.

182. Functions of imaginary variables are in general imaginary quantities capable of expression in the form

$$a + bi,$$

in which $i = \sqrt{(-1)}$.

The transformation of a function of an imaginary variable to the above form may often be effected by developing it into a series, and separating the real and imaginary parts.

Thus, to transform the function ϵ^{ix}, we put ix for x in the exponential series; since $i = \sqrt{(-1)}$ gives

$$i^2 = -1, \qquad i^3 = -i, \qquad i^4 = 1, \text{ etc.,}$$

the result may be written in the form

$$\epsilon^{ix} = 1 + ix - \frac{x^2}{1\cdot 2} - \frac{ix^3}{1\cdot 2\cdot 3} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} + \cdots,$$

or
$$\epsilon^{ix} = \left[1 - \frac{x^2}{1\cdot 2} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} - \cdots\right]$$
$$+ i\left[x - \frac{x^3}{1\cdot 2\cdot 3} + \frac{x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \cdots\right].$$

The series in brackets are the developments of $\cos x$ and of $\sin x$ respectively [see Art. 167]; we have therefore, by substitution,

$$\epsilon^{ix} = \cos x + i \sin x. \qquad \qquad (1)$$

If the independent variable be of the form $a + bi$, the development of the function will contain powers of $a + bi$, and

may therefore, by expanding these powers, be put in the form $a + bi$.

183. Every imaginary quantity may be expressed in the form $\rho(\cos\theta + i\sin\theta)$; for, if we put

$$a + bi = \rho(\cos\theta + i\sin\theta), \quad . \quad . \quad . \quad . \quad . \quad (1)$$

the equations, $a = \rho\cos\theta$ and $b = \rho\sin\theta$, . . . (2)

give $\rho = \sqrt{(a^2 + b^2)}$, and $\tan\theta = \frac{b}{a}$. . . . . (3)

Equations (3) always give real values for ρ and θ. The positive quantity ρ is called the *modulus*, and the angle θ is called the *argument*. The latter admits of but one value between 0 and 2π, the quadrant in which it terminates being determined by the signs of a and b [see equations (2)]; denoting this value by θ', all the values of θ are included in

$$\theta = 2k\pi + \theta',$$

k denoting zero or any integer.

184. The imaginary quantity $a + bi$ can be expressed in the form $\rho\varepsilon^{i\theta}$; for by equation (1), Art. 182, we can write

$$a + bi = \rho(\cos\theta + i\sin\theta) = \rho\varepsilon^{i\theta} \quad . \quad . \quad . \quad . \quad (1)$$

in which ρ and θ are determined as in the preceding article.

From equation (1) we have

$$\log(a + bi) = \log\rho + i\theta = \log\rho + i(\theta' + 2k\pi), \quad . \quad . \quad (2)$$

hence every imaginary quantity has an unlimited number of logarithms; the value of the logarithm obtained by putting $k = 0$ is regarded as the *primary value*.

Putting $b = 0$ we have, when a is positive (see Art. 183),

$$\rho = a \quad \text{and} \quad \theta' = 0 \quad \therefore \quad \log a = \log a + 2k\pi i;$$

whence it follows that $\log a$ may be regarded as a multiple-valued function having but one real value; viz., the primary value as defined above.

To obtain the logarithm of a real negative quantity we put $b = 0$ and $-a$ for a; whence $\rho = a$ and $\theta' = \pi$.

$$\therefore \qquad \log(-a) = \log a + (2k + 1)\pi i,$$

the primary value being $\log a + \pi i$.

185. By changing the sign of i in equation (1) we obtain

$$a - bi = \rho \varepsilon^{-i\theta},$$

in which the modulus is unchanged while the sign of the argument is reversed. $a + bi$ and $a - bi$ are called *conjugate imaginary expressions.* Their sum is the real quantity $2a$, and their product is the real positive quantity $a^2 + b^2$.

Hyperbolic Functions.

186. Exponential expressions for $\sin x$ and $\cos x$ may be derived from equation (1), Art. 182; thus, putting $-x$ for x in

$$\varepsilon^{ix} = \cos x + i \sin x,$$

we have
$$\varepsilon^{-ix} = \cos x - i \sin x;$$

whence
$$\cos x = \tfrac{1}{2}(\varepsilon^{ix} + \varepsilon^{-ix}), \quad \ldots \ldots \quad (1)$$

and
$$\sin x = \frac{1}{2i}(\varepsilon^{ix} - \varepsilon^{-ix}). \quad \ldots \ldots \quad (2)$$

Again, putting ix for x in these expressions for $\sin x$ and $\cos x$, we have

$$\cos ix = \tfrac{1}{2}(\varepsilon^x + \varepsilon^{-x}), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\sin ix = \tfrac{1}{2}i(\varepsilon^x - \varepsilon^{-x}). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The value of $\cos ix$ is real and is called the *hyperbolic cosine* of x, and the real factor of the expression for $\sin ix$ is called the *hyperbolic sine* of x. These real quantities are generally denoted by $\cosh x$ and $\sinh x$, and the ratio $\dfrac{\sinh x}{\cosh x}$ by $\tanh x$. Thus, we have

$$\cosh x = \tfrac{1}{2}(\varepsilon^x + \varepsilon^{-x})$$

$$\sinh x = \tfrac{1}{2}(\varepsilon^x - \varepsilon^{-x})$$

$$\tanh x = \frac{\varepsilon^x - \varepsilon^{-x}}{\varepsilon^x + \varepsilon^{-x}}.$$

The reciprocals of these quantities are denoted by $\operatorname{sech} x$, $\operatorname{cosech} x$, and $\coth x$, respectively.

Formulas involving these functions and presenting a remarkable analogy to the trigonometric formulas may readily be deduced. (See Ex. XXV, 7, 8, 9, 10.)

De Moivre's Theorem.

187. Putting mx for x in equation (1), Art. 182, we have

$$\varepsilon^{imx} = \cos mx + i \sin mx\,;$$

and, since

$$\varepsilon^{imx} = (\varepsilon^{ix})^m,$$

$$\cos mx + i \sin mx = (\cos x + i \sin x)^m. \quad . \quad . \quad . \quad . \quad (1)$$

This result is known as *De Moivre's Theorem*, and is understood to mean that, when the second member of the equation is ex-

panded, the value of the real part will be $\cos mx$, and that of the real factor of i will be $\sin mx$.

188. Expanding the second member of equation (1) by the binomial theorem, we obtain the expansions for $\sin mx$ and $\cos mx$ in terms of $\sin x$ and $\cos x$; the number of terms in each of these expansions will be finite when m is a positive integer.

Thus, putting $m = 3$ in equation (1), we obtain

$$\cos 3x + i \sin 3x = \cos^3 x + 3\cos^2 x \cdot i \sin x - 3 \cos x \cdot \sin^2 x - i \sin^3 x.$$

Whence $$\cos 3x = \cos^3 x - 3 \cos x \sin^2 x,$$

and $$\sin 3x = 3 \cos^2 x \sin x - \sin^3 x.$$

Multiple Values of the n*th Roots of Real and Imaginary Quantities.*

189. Expressions for the roots of quantities in the form

$$\rho(\cos\theta + i \sin\theta)$$

may be derived by means of De Moivre's theorem. Putting $\frac{1}{n}$ for m in equation (1), Art. 187, and employing the general value of θ given in Art. 183, we obtain

$$[\rho(\cos\theta + i \sin\theta)]^{\frac{1}{n}} = \rho^{\frac{1}{n}}\left(\cos\frac{2k\pi + \theta'}{n} + i \sin\frac{2k\pi + \theta'}{n}\right). \quad . \quad (1)$$

When n is an integer, by giving k the values $0, 1, 2, \ldots (n-1)$, we get n values of the arc $\frac{2k\pi + \theta'}{n}$ less than 2π; every other value of this arc will differ from one of these by a multiple of 2π. The second member of equation (1) will therefore have n and only n different values.

190. Making $\rho = 1$ and $\theta' = 0$, we have for the values of the nth root of unity the n values of the expression

$$\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The real root, unity, is obtained by putting $k = 0$. The imaginary root corresponding to $k = 2$ is, by De Moivre's theorem, the square of the one that corresponds to $k = 1$. If the latter is denoted by α, the n roots may be expressed thus,—

$$1, \ \alpha, \ \alpha^2, \ \alpha^3 \ldots . \ \alpha^{n-1}.$$

The roots α and α^{n-1} are conjugate roots; for

$$\alpha^{n-1} = \cos\frac{2(n-1)\pi}{n} + i\sin\frac{2(n-1)\pi}{n} = \cos\frac{2\pi}{n} - i\sin\frac{2\pi}{n},$$

whence (compare Art. 185)

$$\alpha + \alpha^{n-1} = 2\cos\frac{2\pi}{n}, \qquad \text{and} \qquad \alpha\,\alpha^{n-1} = 1.$$

In like manner all the other roots occur in conjugate pairs, with the exception of the real root 1 corresponding to $k = 0$ when n is odd, and of the real roots 1 and -1 corresponding to $k = 0$ and $k = \frac{1}{2}n$ when n is even.

The Resolution of Certain Expressions into Factors by means of De Moivre's Theorem.

191. The expressions derived in the preceding article being roots of the equation

$$x^n - 1 = 0,$$

we have

$$x^n - 1 = (x-1)\left(x - \cos\frac{2\pi}{n} - i\sin\frac{2\pi}{n}\right)\cdots$$

$$\left(x - \cos\frac{2(n-1)\pi}{n} - i\sin\frac{2(n-1)\pi}{n}\right). \quad . \quad . \quad . \quad (1)$$

The product of the factors corresponding to each pair of conjugate roots is a real quadratic factor. Thus,

$$(x-\alpha)(x-\alpha^{n-1}) = x^2 - (\alpha+\alpha^{n-1})x + \alpha^n = x^2 - 2x\cos\frac{2\pi}{n} + 1.$$

Combining in this way the conjugate factors we have, when n is odd,

$$x^n - 1 = (x-1)\left(x^2 - 2x\cos\frac{2\pi}{n} + 1\right)\cdots$$
$$\left(x^2 - 2x\cos\frac{n-1}{n}\pi + 1\right), \quad . \quad . \quad . \quad (2)$$

and when n is even

$$x^n - 1 = (x-1)\left(x^2 - 2x\cos\frac{2\pi}{n} + 1\right)\cdots$$
$$\left(x^2 - 2x\cos\frac{n-2}{n}\pi + 1\right)(x+1). \quad . \quad . \quad . \quad (3)$$

192. *To resolve* $x^{2n} - 2x^n\cos\theta + 1$ *into factors,* n being an integer, we solve the equation

$$x^{2n} - 2x^n\cos\theta + 1 = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

From (1) we obtain

$$x^n = \cos\theta \pm i\sin\theta.$$

Hence, expressing the values of x by means of equation (1), Art. 189, we have

$$x = \cos\frac{2k\pi+\theta}{n} \pm i\sin\frac{2k\pi+\theta}{n}, \quad . \quad . \quad . \quad (2)$$

the values of k being 0, 1, 2, ... $(n-1)$. The product of the factors corresponding to the two conjugate roots determined

by a single value of k in equation (2) reduces to the real quadratic expression

$$x^2 - 2x\cos\frac{2k\pi + \theta'}{n} + 1.$$

Hence we have

$$x^{2n} - 2x^n\cos\theta + 1 = \left(x^2 - 2x\cos\frac{\theta'}{n} + 1\right)\left(x^2 - 2x\cos\frac{2\pi+\theta'}{n} + 1\right)\cdots$$

$$\left(x^2 - 2x\cos\frac{2(n-1)\pi + \theta'}{n} + 1\right). \quad . \quad . \quad (3)$$

The Sine and the Cosine Expressed as Continued Products.

193. Putting $x = 1$ in equation (3), the first member becomes $2(1 - \cos\theta)$, and each factor in the second member takes a similar form. The *positive* square root of $2(1 - \cos\theta)$ is $2\sin\frac{1}{2}\theta'$, since $\frac{1}{2}\theta'$ is between 0 and π. Taking in like manner the square root of each factor in the second member, we obtain

$$2\sin\tfrac{1}{2}\theta' = 2^n\sin\frac{\theta'}{2n}\sin\frac{2\pi+\theta'}{2n}\sin\frac{4\pi+\theta'}{2n}\cdots\sin\frac{2(n-1)\pi+\theta'}{2n}. \ (1)$$

The above equation is however true for all values of θ; for, if we add 2π to θ', the first member changes sign; but in the second member the first factor assumes the present form of the second factor, the second that of the third, and so on; finally the last factor becomes equal to the present value of the first factor with its sign changed; therefore the second member also changes sign. Hence θ' may have any value, and putting θ in place of $\frac{1}{2}\theta'$ we have

$$\sin\theta = 2^{n-1}\sin\frac{\theta}{n}\sin\left(\frac{\pi}{n} + \frac{\theta}{n}\right)\sin\left(\frac{2\pi}{n} + \frac{\theta}{n}\right)\cdots$$

$$\sin\left(\frac{(n-1)\pi}{n} + \frac{\theta}{n}\right). \quad . \quad . \quad . \quad (2)$$

The last factor may be written in the form

$$\sin\left(\pi - \frac{\pi}{n} + \frac{\theta}{n}\right) = \sin\left(\frac{\pi}{n} - \frac{\theta}{n}\right),$$

hence for the product of the second and last factors we have

$$\sin\left(\frac{\pi}{n} + \frac{\theta}{n}\right)\sin\left(\frac{\pi}{n} - \frac{\theta}{n}\right) = \sin^2\frac{\pi}{n} - \sin^2\frac{\theta}{n}.$$

In a similar manner the third factor and the last but one may be combined; therefore, if n is an odd number, we have

$$\sin\theta = 2^{n-1}\sin\frac{\theta}{n}\left(\sin^2\frac{\pi}{n} - \sin^2\frac{\theta}{n}\right)\left(\sin^2\frac{2\pi}{n} - \sin^2\frac{\theta}{n}\right)\cdots$$

$$\left(\sin^2\frac{n-1}{2}\frac{\pi}{n} - \sin^2\frac{\theta}{n}\right) \quad . \quad . \quad . \quad . \quad (3)$$

Dividing this equation by $\sin\theta$ and then making $\theta = 0$, we have

$$1 = \frac{2^{n-1}}{n}\sin^2\frac{\pi}{n}\sin^2\frac{2\pi}{n}\cdots\sin^2\frac{(n-1)\pi}{2n},$$

and dividing equation (3) by the last equation, we have

$$\sin\theta = n\sin\frac{\theta}{n}\left[1 - \frac{\sin^2\frac{\theta}{n}}{\sin^2\frac{\pi}{n}}\right]\left[1 - \frac{\sin^2\frac{\theta}{n}}{\sin^2\frac{2\pi}{n}}\right]\cdots\left[1 - \frac{\sin^2\frac{\theta}{n}}{\sin^2\frac{(n-1)\pi}{2n}}\right].$$

Finally in the above equation if θ remains fixed, and n increases without limit we have, on evaluation,

$$\sin\theta = \theta\left[1 - \frac{\theta^2}{\pi^2}\right]\left[1 - \frac{\theta^2}{2^2\pi^2}\right]\left[1 - \frac{\theta^2}{3^2\pi^2}\right]\cdots, \quad . \quad (4)$$

the number of factors being unlimited.

194. A similar expression for $\cos\theta$ may be derived from equation (4) of the preceding article, by means of the formula

$$\cos\theta = \frac{\sin 2\theta}{2\sin\theta};$$

whence

$$\cos\theta = \frac{2\theta\left(1-\frac{4\theta^2}{\pi^2}\right)\left(1-\frac{4\theta^2}{4\pi^2}\right)\left(1-\frac{4\theta^2}{9\pi^2}\right)\left(1-\frac{4\theta^2}{16\pi^2}\right)\cdots}{2\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{4\pi^2}\right)\cdots},$$

and removing common factors, we have

$$\cos\theta = \left[1-\frac{4\theta^2}{\pi^2}\right]\left[1-\frac{4\theta^2}{9\pi^2}\right]\left[1-\frac{4\theta^2}{25\pi^2}\right]\cdots \qquad (5)$$

By expanding the continued products in equations (4) and (5) and equating the coefficients of given powers of θ to the corresponding coefficients in the expansion given in Art. 167, certain numerical series involving powers of π may be derived. Thus, equating the coefficients of θ^3 in the expansions of $\sin\theta$, we derive

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{6}.$$

Bernoulli's Numbers.

195. A series of numbers, first employed by James Bernoulli in 1687, present themselves in the expansions of certain functions, among which is the function

$$\tfrac{1}{2}x\,\frac{\varepsilon^x+1}{\varepsilon^x-1}.$$

This expression is easily shown to be an even function; hence if we make the transformation

$$\tfrac{1}{2}x\frac{\varepsilon^x + 1}{\varepsilon^x - 1} = \tfrac{1}{2}x + \frac{x}{\varepsilon^x - 1}, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and write
$$y = \frac{x}{\varepsilon^x - 1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

the development of y will contain no terms involving odd powers of x except the term $-\tfrac{1}{2}x$.

196. To determine the values of y and of its derivatives when $x = 0$, we have from equation (2)

$$\varepsilon^x y - y = x, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and differentiating
$$\varepsilon^x \frac{dy}{dx} + \varepsilon^x y - \frac{dy}{dx} = 1. \quad . \quad . \quad . \quad . \quad (4)$$

By applying Leibnitz' theorem (Art. 88) to equation (3), we derive, for values of n greater than unity,

$$\varepsilon^x \frac{d^n y}{dx^n} + n\varepsilon^x \frac{d^{n-1} y}{dx^{n-1}} + \frac{n(n-1)}{1 \cdot 2}\varepsilon^x \frac{d^{n-2} y}{dx^{n-2}} \cdots + \varepsilon^x y - \frac{d^n y}{dx^n} = 0; \quad (5)$$

putting $x = 0$ in (4), we have $y_0 = 1$; therefore equation (5) becomes, when $x = 0$,

$$n \frac{d^{n-1}y}{dx^{n-1}}\bigg]_0 + \frac{n(n-1)}{1 \cdot 2} \cdot \frac{d^{n-2}y}{dx^{n-2}}\bigg]_0 \cdots + n\frac{dy}{dx}\bigg]_0 + 1 = 0. \quad . \quad (6)$$

From equation (6) the values of the derivatives may be found by putting $n = 2, 3, 4$, etc. Thus making $n = 2$ we have

$$\frac{dy}{dx}\bigg]_0 = -\frac{1}{2};$$

but, since the development of y contains no odd powers of x except the first, the remaining odd derivatives vanish. The numerical values of the even derivatives are Bernoulli's Numbers; the notation adopted being

$$B_n = -(-1)^n \frac{d^{2n}y}{dx^{2n}}\bigg]_0, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

because, as will be shown in Art. 199, the values of

$$B_1, \quad B_2, \quad B_3, \quad \text{etc.},$$

thus defined, are all positive. The development of y may therefore be written

$$y = \frac{x}{e^x - 1} = 1 - \frac{x}{2} + B_1 \frac{x^2}{1 \cdot 2} - B_2 \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4}$$

$$+ B_3 \frac{x^6}{1 \cdot 2 \cdots 6} - B_4 \frac{x^8}{1 \cdot 2 \cdots 8} + \cdots \quad . \quad . \quad . \qquad (8)$$

Putting $2n + 1$ in place of n in equation (6), adopting the notation of equation (7), and introducing the values of the derivatives already determined, we obtain

$$-(-1)^n(2n+1)B_n + (-1)^n \frac{(2n+1)\,2n(2n-1)}{1 \cdot 2 \cdot 3} B_{n-1} \cdots$$

$$+ \frac{(2n+1)2n}{1 \cdot 2} B_1 - \frac{2n-1}{2} = 0,$$

and, solving for B_n,

$$B_n = n\frac{2n-1}{3} B_{n-1} - n\frac{(2n-1)(2n-2)(2n-3)}{3 \cdot 4 \cdot 5} B_{n-2} + \cdots$$

$$+ (-1)^n n B_1 - (-1)^n \frac{2n-1}{2(2n+1)}.$$

Substituting for n the successive values 1, 2, 3, etc., we derive

$$B_1 = \tfrac{1}{6}, \quad B_2 = \tfrac{1}{30}, \quad B_3 = \tfrac{1}{42}, \quad B_4 = \tfrac{1}{30}, \quad B_5 = \tfrac{5}{66},$$

$$B_6 = \tfrac{691}{2730}, \quad B_7 = \tfrac{7}{6}, \quad B_8 = \tfrac{3617}{510}.$$

The Development of $\theta \cot \theta$.

197. The development of the even function $\frac{1}{2}x\dfrac{\varepsilon^x + 1}{\varepsilon^x - 1}$ differs from that of y in equation (8) of the preceding article only in the absence of the term containing the first power of x (see Art. 195). This function remains real when ix is substituted for x; for we have

$$\frac{1}{2}ix\frac{\varepsilon^{ix} + 1}{\varepsilon^{ix} - 1} = 1 - B_1\frac{x^2}{1\cdot 2} - B_2\frac{x^4}{1\cdot 2\cdot 3\cdot 4} - B_3\frac{x^6}{1\cdot 2\cdots 6} - \cdots$$

But (see equations (1) and (2), Art. 186),

$$\frac{1}{2}ix\frac{\varepsilon^{ix} + 1}{\varepsilon^{ix} - 1} = \frac{1}{2}ix\frac{\varepsilon^{\frac{1}{2}ix} + \varepsilon^{-\frac{1}{2}ix}}{\varepsilon^{\frac{1}{2}ix} - \varepsilon^{-\frac{1}{2}ix}} = \frac{1}{2}ix\frac{2\cos\frac{1}{2}x}{2i\sin\frac{1}{2}x} = \frac{1}{2}x\cot\frac{1}{2}x;$$

hence putting θ for $\frac{1}{2}x$

$$\theta\cot\theta = 1 - B_1\frac{2^2\theta^2}{1\cdot 2} - B_2\frac{2^4\theta^4}{1\cdot 2\cdot 3\cdot 4} - B_3\frac{2^6\theta^6}{1\cdot 2\cdots 6} - \cdots \quad . \quad (1)$$

198. A series for $\theta\cot\theta$ in which the coefficients of the powers of θ assume a different form is obtained from equation (4), Art. 193, by taking logarithmic derivatives; thus,

$$\cot\theta = \frac{1}{\theta} - \frac{\dfrac{2\theta}{\pi^2}}{1 - \dfrac{\theta^2}{\pi^2}} - \frac{\dfrac{2\theta}{2^2\pi^2}}{1 - \dfrac{\theta^2}{2^2\pi^2}} - \frac{\dfrac{2\theta}{3^2\pi^2}}{1 - \dfrac{\theta^2}{3^2\pi^2}} - \cdots;$$

whence $\theta\cot\theta = 1 - \dfrac{2\theta^2}{\pi^2}\left(1 - \dfrac{\theta^2}{\pi^2}\right)^{-1} - \dfrac{2\theta^2}{2^2\pi^2}\left(1 - \dfrac{\theta^2}{2^2\pi^2}\right)^{-1} - \cdots$

The expansion of each term after the first in the second member is of the form

$$-\frac{2\theta^2}{k^2\pi^2}\left(1 - \frac{\theta^2}{k^2\pi^2}\right)^{-1} = -2\frac{\theta^2}{k^2\pi^2}\left(1 + \frac{\theta^2}{k^2\pi^2} + \frac{\theta^4}{k^4\pi^4} + \cdots\right),$$

in which k has the successive values 1, 2, 3, etc.

Substituting these expansions and collecting the coefficients of the powers of θ, we have

$$\theta \cot \theta = 1 - 2\frac{\theta^2}{\pi^2}\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$
$$- 2\frac{\theta^4}{\pi^4}\left[1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots\right] - 2\frac{\theta^6}{\pi^6}\left[1 + \frac{1}{2^6} + \cdots\right].$$

199. Comparing this expansion with (1) of Art. 197, we obtain

$$B_1 \frac{2^2}{1\cdot 2} = \frac{2}{\pi^2}\left[1 + \frac{1}{2^2} + \frac{1}{3^2}\cdots\right],$$

$$B_2 \frac{2^4}{1\cdot 2\cdot 3\cdot 4} = \frac{2}{\pi^4}\left[1 + \frac{1}{2^4} + \frac{1}{3^4}\cdots\right] \text{ etc.,}$$

and in general $B_n \dfrac{2^{2n}}{1\cdot 2\cdots 2n} = \dfrac{2}{\pi^{2n}}\left[1 + \dfrac{1}{2^{2n}} + \dfrac{1}{3^{2n}}\cdots\right].$

Hence we have

$$1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}}\cdots = \frac{(2\pi)^{2n} B_n}{2\cdot 1\cdot 2\cdots 2n}, \quad . \quad . \quad . \quad . \quad (1)$$

which serves to determine the sum of an arithmetical series of the above form in terms of the Bernoullian numbers. For instance when $n = 1$, we have the result already obtained in Art. 194, and when $n = 2$, we have

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90}.$$

Equation (1) may also be written in the form

$$B_n = \frac{2\cdot 1\cdot 2\cdots 2n}{(2\pi)^{2n}}\left[1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}}\cdots\right],$$

which shows that Bernoulli's numbers are all positive; and also

that they increase rapidly with n; for, as n increases B_n approaches but always exceeds

$$\frac{2 \cdot 1 \cdot 2 \cdot \cdot \cdot 2n}{(2\pi)^{2n}}.$$

Examples XXV.

1. Prove that

$$\tan^{-1} ix = \tfrac{1}{2} i \log \frac{1+x}{1-x}.$$

See Art. 171 *and equation* (3), *Art.* 164. *The real factor in this expression for* $\tan^{-1} ix$ *is the inverse hyperbolic tangent of* x.

2. Prove that

$$\sin^{-1} ix = i \log [x + \sqrt{(1 + x^2)}],$$

by means of the results obtained in Ex. XXIV, 9 and 10.

The real factor in this expression for $\sin^{-1} ix$ *is the inverse hyperbolic sine of* x.

3. Prove that

$$[\sqrt{(1 - x^2)} + ix]^m = \cos [m \sin^{-1} x] + i \sin [m \sin^{-1} x] = \varepsilon^{mi \sin^{-1} x}.$$

See Ex. XXIV, 11, *and Art.* 177.

4. Find the primary value of $\log i$ (see Art. 184.)

$$\log i = \tfrac{1}{2} i\pi.$$

5. Prove by means of the result obtained in the preceding example that

$$\sqrt[i]{i} = \sqrt{\varepsilon^{\pi}}.$$

6. Derive the fundamental formulas of Trigonometry by transforming the identity $\varepsilon^{iu} \varepsilon^{iv} = \varepsilon^{iu+iv}$, by means of equation (1), Art. 182.

7. Prove the relations :—

$$\cosh^2 x - \sinh^2 x = 1, \quad \text{and} \quad \operatorname{sech}^2 x = 1 - \tanh^2 x.$$

8. Prove the relations :—

$$\sinh 2x = 2 \sinh x \cosh x, \quad \text{and} \quad \cosh 2x = \cosh^2 x + \sinh^2 x.$$

9. Prove the formulas :—

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y,$$

and

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y.$$

10. Prove the formulas :—

$$d(\sinh x) = \cosh x\, dx,\quad d(\cosh x) = \sinh x\, dx,\quad d(\tanh x) = (\operatorname{sech} x)^2 dx,$$

$$d(\operatorname{sech} x) = -\operatorname{sech} x \tanh x\, dx,\quad d(\operatorname{cosech} x) = -\operatorname{cosech} x \coth x\, dx,$$

$$d(\coth x) = -(\operatorname{cosech} x)^2\, dx.$$

11. Find expressions for $\sin 5x$ and $\cos 5x$ by means of De Moivre's theorem.

$$\sin 5x = 5\cos^4 x \sin x - 10\cos^2 x \sin^3 x + \sin^5 x.$$

$$\cos 5x = \cos^5 x - 10\cos^3 x \sin^2 x + 5\cos x \sin^4 x.$$

12. Find expressions for $\sin mx$ and $\cos mx$ by means of De Moivre's theorem.

$$\cos mx = \cos^m x - \frac{m(m-1)}{1\cdot 2}\cos^{m-2}x \sin^2 x + \cdots$$

$$\sin mx = m\cos^{m-1}x \sin x - \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}\cos^{m-3}x \sin^3 x + \cdots$$

13. Find the sixth roots of unity.

The roots are $\pm 1,\quad \pm\left(\frac{1}{2} \pm \frac{i}{2}\sqrt{3}\right)$.

14. Find the fourth roots of -1.

The roots are $\pm \frac{1}{2}\sqrt{2}(1 \pm i)$.

15. Find the cube roots of i.

The roots are $-i,\quad \pm\frac{1}{2}\sqrt{3} + \frac{1}{2}i$.

16. Resolve $x^n + 1$ into factors.

When n is even,

$$x^n + 1 = \left(x^2 - 2x\cos\frac{\pi}{n} + 1\right)\cdots\left(x^2 - 2x\cos\frac{n-2}{n}\pi + 1\right)(x+1).$$

$$\left(x^2 - 2x\cos\frac{n-1}{n}\pi + 1\right);$$

when n is odd,

$$x^n + 1 = \left(x^2 - 2x\cos\frac{\pi}{n} + 1\right)\cdots\left(x^2 - 2x\cos\frac{n-2}{n}\pi - 1\right)(x+1).$$

17. Express the hyperbolic sine in the form of an infinite product by putting ix for θ in equation (4) Art. 193.

$$\sinh x = x\left(1 + \frac{x^2}{\pi^2}\right)\left(1 + \frac{x^2}{2^2\pi^2}\right)\left(1 + \frac{x^2}{3^2\pi^2}\right)\cdots$$

18. Express the hyperbolic cosine in the form of an infinite product by putting ix for θ in the expression for $\cos x$ derived in Art. 194.

$$\cosh x = \left(1 + \frac{4x^2}{\pi^2}\right)\left(1 + \frac{4x^2}{3^2\pi^2}\right)\left(1 + \frac{4x^2}{5^2\pi^2}\right)\cdots$$

19. By means of equation (8), Art. 196, and the relation

$$\frac{x}{\varepsilon^x + 1} = \frac{x}{\varepsilon^x - 1} - \frac{2x}{\varepsilon^{2x} - 1},$$

find the development of $\dfrac{x}{\varepsilon^x + 1}$.

$$\frac{x}{\varepsilon^x + 1} = \frac{x}{2} - B_1\frac{2^2 - 1}{1\cdot 2}\cdot x^2 + B_2\frac{2^4 - 1}{1\cdot 2\cdot 3\cdot 4}x^4 - B_3\frac{2^6 - 1}{1\cdot 2\cdots 6}x^6 + \cdots$$

20. From the result obtained in the preceding example, derive the development of $\dfrac{\varepsilon^x - 1}{\varepsilon^x + 1}$.

$$\frac{\varepsilon^x - 1}{\varepsilon^x + 1} = \frac{2(2^2 - 1)B_1}{1\cdot 2}x - \frac{2(2^4 - 1)B_2}{1\cdot 2\cdot 3\cdot 4}x^3 + \frac{2(2^6 - 1)B_3}{1\cdot 2\cdots 6}x^5 - \cdots,$$

or, employing the values of B_1, B_2, etc., given in Art. 196,

$$\frac{\varepsilon^x - 1}{\varepsilon^x + 1} = \frac{x}{2} - \frac{x^3}{24} + \frac{x^5}{240} - \frac{17x^7}{40320} + \frac{31x^9}{725760} - \cdots$$

21. By means of the identity

$$\tan x = \cot x - 2\cot 2x,$$

derive the development of $\tan x$ from equation (1), Art. 197.

$$\tan x = B_1\frac{2^2(2^2 - 1)}{1\cdot 2}x + B_2\frac{2^4(2^4 - 1)}{1\cdot 2\cdot 3\cdot 4}x^3 + B_3\frac{2^6(2^6 - 1)}{1\cdot 2\cdots 6}x^5 + \cdots$$

22. By means of the identity

$$\operatorname{cosec} x = \cot\frac{x}{2} - \cot x,$$

derive the development of cosec x from equation (1), Art. 197.

$$\operatorname{cosec} x = \frac{1}{x} + B_1\frac{2(2-1)}{1\cdot 2}x + B_2\frac{2(2^3-1)}{1\cdot 2\cdot 3\cdot 4}x^3 + \cdots$$

23. By putting $\theta = \pi x$ and taking logarithmic derivatives derive from equation (4), Art. 193, the series,

$$\pi\cot\pi x = \frac{1}{x} + \frac{1}{x-1} + \frac{1}{x+1} + \frac{1}{x-2} + \frac{1}{x+2} + \frac{1}{x-3} + \cdots$$

24. By a method similar to that employed in the preceding example, derive the series

$$\frac{\pi}{2}\tan\pi x = \frac{1}{1-2x} - \frac{1}{1+2x} + \frac{1}{3-2x} - \frac{1}{3+2x} + \cdots$$

25. By taking derivatives of the series obtained in example 23 and putting $x = \frac{1}{4}$, derive the numerical series :—

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots,$$

$$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots,$$

$$\frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots.$$

CHAPTER VIII.

CURVE TRACING.

XXVI.

Equations in the Form y = f(x).

200. WHEN a curve given by its equation is to be traced, it is necessary to determine its general form especially at such points as present any peculiarity, and also the nature of those branches of the curve, if there be any, which are unlimited in extent.

The general mode of procedure, when the equation can be put in either of the forms, $y = f(x)$ or $x = \phi(y)$, is indicated in the following examples.

Asymptotes Parallel to the Coordinate Axes.

201. *Example* 1. $a^2y - x^2y = a^3$ . . . . (1)
Solving for y, we obtain

$$y = \frac{a^3}{a^2 - x^2}. \qquad (2)$$

When $x = 0, y = a$. Numerically equal positive and negative values of x give the same values for y; the curve is therefore symmetrical with reference to the axis of y. As x increases

from zero, y increases until the denominator, $a^2 - x^2$, becomes zero, when y becomes infinite; this occurs when $x = \pm a$.

Draw the straight lines $x = \pm a$. These are lines to which the curve approaches indefinitely, for we may assign values to x as near as we please to $+a$ or to $-a$, thus determining points of the curve as near as we please to the straight lines $x = a$ and $x = -a$. Such lines are called *asymptotes* to the curve.

When x passes the value a, y becomes negative and decreases numerically, approaching the value zero as x increases indefinitely. Hence there is a branch of the curve below the axis of x to which the lines $x = a$ and $y = 0$ are asymptotes.

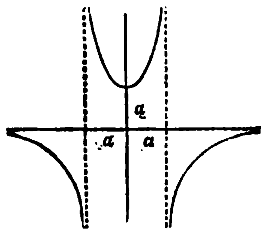


FIG. 16.

The general form of the curve is indicated in Fig. 16.

The point $(0, a)$ evidently corresponds to a minimum ordinate.

202. *Example* 2. $a^2x = y(x-a)^2$. . . . . (1)
Solving for y,

$$y = \frac{a^2x}{(x-a)^2}. \qquad (2)$$

When x is zero, y is zero; y increases as x increases until $x = a$, when y becomes infinite. Hence

$$x = a$$

is the equation of an asymptote. When x passes the value a, y does not change sign, but remains positive, and as x increases y diminishes, approaching zero as x increases indefinitely.

Examining now the values of y which correspond to negative values of x, we perceive that, y becoming negative, the branch which passes through the origin continues below the axis of x, and that y approaches zero as the negative value of x increases indefinitely. Hence the general form of the curve is that indicated in Fig. 17.

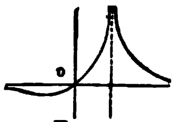


FIG. 17.

203. To determine the direction of the curve at any point, we have

$$\tan\phi = \frac{dy}{dx} = -a^2\frac{a+x}{(x-a)^3}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The direction in which the curve passes through the origin is given by the value of $\tan\phi$ which corresponds to $x = 0$. From (3), we have

$$\left.\frac{dy}{dx}\right]_0 = 1;$$

the inclination of the curve at the origin is therefore 45°.

Minimum Ordinates and Points of Inflexion.

204. To find the minimum ordinate which evidently exists on the left of the axis of y, we put the expression for $\frac{dy}{dx}$ equal to zero, and deduce

$$x = -a.$$

The minimum ordinate is therefore found at the point whose abscissa is $-a$; its value, obtained from equation (2), is $-\frac{1}{4}a$.

A *point of inflexion* is a point at which $\frac{d^2y}{dx^2}$ changes sign (see

Art. 76); in other words, it is a point at which $\tan \phi$ has a maximum or a minimum value. In this case there is evidently a point of inflexion on the left of the minimum ordinate. From equation (3) we derive

$$\frac{d^2y}{dx^2} = 2a^3 \frac{x + 2a}{(x - a)^4},$$

putting this expression equal to zero to determine the abscissa, and deducing the corresponding ordinate from (2), we obtain, for the coordinates of the point of inflexion,

$$x = -2a, \quad \text{and} \quad y = -\tfrac{8}{9}a.$$

Oblique Asymptotes.

205. *Example* 3. $x^3 - 2x^2y - 2x^2 - 8y = 0. \quad . \quad . \quad . \quad . \quad (1)$
Solving this equation for y, we have

$$y = \frac{x^2}{2} \cdot \frac{x - 2}{x^2 + 4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

It is obvious from the form of equation (2) that the curve meets the axis of x at the two points (0, 0) and (2, 0). Since y is positive only when $x > 2$, the curve lies below the axis of x on the left of the origin, and also between the origin and the point (2, 0), but on the right of this point the curve lies above the axis of x.

206. Developing the second member of equation (2) into an expression involving a fraction whose numerator is lower in degree than its denominator, we have

$$y = \tfrac{1}{2}x - 1 + 2\frac{2 - x}{x^2 + 4}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The fraction in this expression decreases without limit as x increases indefinitely; hence the ordinate of the curve may, by increasing x, be made to differ as little as we please from that of the straight line

$$y = \tfrac{1}{2}x - 1.$$

This line is, therefore, an asymptote.

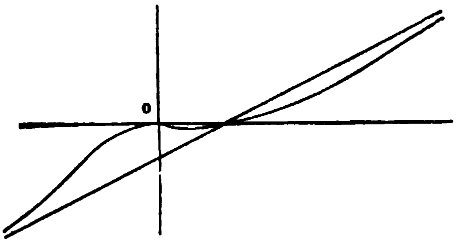


Fig. 18.

The fraction $\dfrac{2-x}{x^2+4}$ is positive for all values of x less than 2, negative for all values of x greater than 2, and does not become infinite. The curve, therefore, lies above the asymptote on the left of the point (2, 0), and below it on the right of this point, as represented in Fig. 18.

207. There is evidently a minimum ordinate between the origin and the point (2, 0).

We obtain from equation (2)

$$\frac{dy}{dx} = \frac{x}{2} \cdot \frac{x^3 + 12x - 16}{(x^2+4)^2}, \quad \ldots \ldots \quad (4)$$

and

$$\frac{d^2y}{dx^2} = 4 \cdot \frac{-x^3 + 6x^2 + 12x - 8}{(x^2+4)^3}. \quad \ldots \quad (5)$$

Putting $\dfrac{dy}{dx} = 0$, we obtain $x = 0$ and $x = 1.19$ nearly, the only real roots; the abscissa corresponding to the minimum ordinate is therefore 1.19, the value of the ordinate being about -0.11. The root zero corresponds to a maximum ordinate at the origin.

Putting $\dfrac{d^2y}{dx^2} = 0$, we obtain the three roots $x = -2$, and $x = 2(2 \pm \sqrt{3})$; the corresponding ordinates are obtained from equation (3). There are, therefore, three points of inflexion, one situated at the point $(-2, -1)$, and the others near the points $(0.54, -0.05)$, and $(7.46, 2.55)$.

The inclination of the curve is determined by means of equation (4) to be $\tan^{-1}\frac{1}{4}$ at the point $(2, 0)$, and $\tan^{-1}\frac{5}{4}$ at the left-hand point of inflexion.

Curvilinear Asymptotes.

208. *Example* 4. $x^3 - xy + 1 = 0.$ . . . . (1)

Solving for y

$$y = \frac{x^3 + 1}{x} = x^2 + \frac{1}{x}. \qquad (2)$$

In this case, on developing y in powers of x, the integral portion of its value is found to contain the second power of x; the fraction approaches zero when x increases indefinitely; hence the ordinate of this curve may be made to differ as little as we please from that of the curve

$$y = x^2. \qquad (3)$$

The parabola represented by this equation is accordingly said to be a *curvilinear asymptote.* It is indicated by the dotted line in Fig. 19.

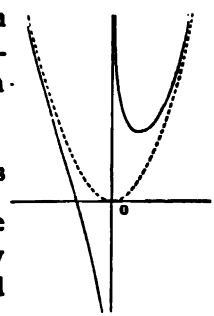


FIG. 19.

209. The sign of the fraction $\frac{1}{x}$ is always the same as that of x, and its value is infinite when x is zero; hence the curve lies below the parabola on the left of the axis of y, and above it on the right, this axis being an asymptote, as indicated in Fig. 19.

Taking derivatives, we obtain

$$\frac{dy}{dx} = 2x - \frac{1}{x^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and

$$\frac{d^2y}{dx^2} = 2\left(1 + \frac{1}{x^3}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

There is a point of inflexion at $(-1, 0)$; the inclination of the curve to the axis of x at this point is $\tan^{-1}(-3)$.

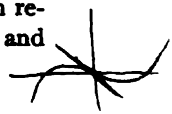
There is a minimum ordinate at the point $(\frac{1}{2}\sqrt[3]{4}, \frac{3}{2}\sqrt[3]{2})$.

This cubic curve is an example of the species called by Newton *the trident*. The characteristic property of a trident is the possession of a parabolic asymptote and a rectilinear asymptote parallel to the axis of the parabola.

Examples XXVI.

(4) + 1. Trace the curve $y = x(x^2 - 1)$.

Since y is an odd function of x, the curve is symmetrical with reference to the origin as a centre. Find the point of inflexion, and the minimum ordinate.



(5) 2. Trace the curve $y^2(x - 1) = x^3$.

The curve has for an asymptote the line $x = 1$; there is a minimum ordinate at $(2, 2)$, and a point of inflexion at $(4, \frac{8}{3}\sqrt{3})$.

(6) 3. Trace the curve $y^3 = x^2(x - a)$, determining its direction at the points at which it meets the axis of x.

The asymptote is found by the method of development, thus

$$y = x\left(1 - \frac{a}{x}\right)^{\frac{1}{3}} = x\left(1 - \frac{a}{3x} - \frac{a^2}{9x^2} - \text{etc.}\right);$$

the equation of the asymptote is therefore

$$y = x - \tfrac{1}{3}a.$$

(7) 4. Trace the curve $x^2 + xy + 2x - y = 0$.

(8) 5. Trace the curve $y^2 = x^2 + x^3$, and find its direction at the origin.

The curve has a maximum ordinate at $(-\frac{2}{3}, \pm \frac{2}{9}\sqrt{3})$. The value of y^2 may be taken as the function whose maximum is required. (See Art. 130.)

6. Trace the curve $y = x^3 - xy$. Find the point of inflexion, the minimum ordinate, and the asymptotes.

The curve has a rectilinear asymptote $x = -1$, and a curvilinear asymptote $y = x^2 - x + 1$. This curve is a trident. (See Art. 209.)

7. Trace the curve $y^2 = x^3 - x^4$.

Both branches of the curve are tangent to the axis of x at the origin.

8. Trace the curve $y^2 - 4y = x^3 + x^2$.

Solving for y, we obtain

$$y = 2 \pm \sqrt{(x^3 + x^2 + 4)} = 2 \pm \sqrt{[(x + 2)(x^2 - x + 2)]}.$$

The factor $x^2 - x + 2$ being always positive, the curve is real on the right of the line $x = -2$.

Find the points at which the curve cuts the axis, and show that the upper branch has a maximum ordinate for $x = -\frac{2}{3}$ and a minimum ordinate for $x = 0$.

9. Trace the curve $(x - 2a)xy = a(x - a)(x - 3a)$.

10. Trace the curve $(x - 2a)xy^2 = a^2(x - a)(x - 3a)$.

11. Trace the curve $y^2 = x^4(1 - x^2)^3$: find all the points at which the tangent is parallel to the axis of x.

12. Trace the curve $6x(1 - x)y = 1 + 3x$.

This curve has a point of inflexion, determined by a cubic having only one real root, which is between -1 and -2. Find the three asymptotes, and the maximum and the minimum ordinate.

13. Trace the curve $x^2y^2 = (x + 2)^2 (1 + x^2)$.

Solving the equation for y, we have

$$y = \pm \frac{x+2}{x} \sqrt{(1 + x^2)} = \pm (2 + x)\left(1 + \frac{1}{x^2}\right)^{\frac{1}{2}}.$$

The asymptotes are $y = x + 2$, $y = -x - 2$, and $x = 0$. The curve has a minimum ordinate corresponding to $x = \sqrt[3]{2}$; the inclination at the point at which it cuts the axis of x is $\tan^{-1}(\pm \frac{1}{2}\sqrt{5})$. There is a point of inflexion corresponding to the abscissa $x = -6.1$ nearly.

14. Trace the curve $(y^2 - b^2)^2 = a^3x$; determine the points at which $\tan\phi$ is infinite; and, denoting by x_0 the abscissa of the point at which the curve cuts the axis of x, show that the abscissa of the two points of inflexion is $\frac{4}{9}x_0$.

15. Trace the curve $y^2(x - a) = x^3 + ax^2$.

Solving for y, we have

$$y = \pm x\sqrt{\left(\frac{x+a}{x-a}\right)} = \pm (x + a)\left(1 - \frac{a^2}{x^2}\right)^{-\frac{1}{2}},$$

or

$$y = \pm (x + a)\left(1 + \frac{a^2}{2x^2} + \text{etc.}\right).$$

The asymptotes are $x = a$, $y = x + a$, and $y = -x - a$. There is a point of inflexion corresponding to $x = -2a$.

16. Trace the curve $y^2x^2 = a^2(x^2 + y^2)$.

The asymptotes are $x = \pm a$, and $y = \pm a$.

17. Trace the curve $x^2(y - x) + xy + x^2 + x + 2 = 0$.

Show that the equation of the oblique asymptote is $y = x - 2$. Find the points in which the curve cuts the axis of x.

18. Trace the curve $(x - k)y = (x - a)(x - b)(x - c)$.

Taking $a > b > c$, trace the curve, first for $k < c$; secondly, for $k > c$ and $< b$. Examine also the intermediate case in which $k = c$.

19. Trace the curve $(x - k)y^2 = (x - a)(x - b)(x - c)$.

Taking $a > b > c$, trace the curve, first for $k < c$; secondly, for $k > c$ and $< b$. Examine also the cases in which $k = c, a = b$, and $a = b = c$.

20. Trace the curve $(y - x^2)^2 - x^5 = 0$.

Both branches of the curve touch the axis of x at the origin. The lower branch cuts the axis of x, has a point of inflexion, and a maximum ordinate.

21. Trace the curve $y^3 - axy - b^2x = 0$.

This curve is a trident (see Art. 209), having a point of inflexion at the origin.

XXVII.

Equations in the Form $f(x, y) = 0$.

210. In this section and the next we illustrate by examples some of the methods which may be employed in tracing algebraic curves, when it is not possible to solve the equation so as to make one coordinate an explicit function of the other.

Example 5. $x^3 - y^3 - x^2 + 2y^2 = 0$. . . . . (1)

The points in which this curve cuts the axes are readily obtained; thus, putting

$$x = 0,$$

we have $$y^2(2 - y) = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

the roots of this equation are 0, 0, and 2; hence the origin and the point B (0, 2) are points of the curve (see Fig. 20). Putting

$$y = 0,$$

we have $$x^2(x - 1) = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

the roots are 0, 0, and 1; hence the point A (1, 0) is a point of the curve.

The Employment of an Auxiliary Variable.

211. The intersections of the curve with the line

$$y = mx$$

are determined by the equation obtained by substituting this value of y in equation (1); viz.,

$$x^3(1 - m^2) - x^2(1 - 2m^2) = 0. \quad . \quad . \quad . \quad . \quad (4)$$

Since x^2 appears as a factor of the first member, $x = 0$ is a double root of this equation, *whatever be the value of* m; hence every line of the form $y = mx$ is said to meet the curve in *two coincident points* at the origin. In fact, there are in this case two branches of the curve passing through the origin, and this furnishes an explanation of the double roots found in equations (2) and (3).

212. The curve may now be traced by means of the following expression for the third root of equation (4), which is the abscissa of P, the only point *distinct from the origin* in which the line $y = mx$ cuts the curve.

Rejecting the factor x^2, we obtain, from equation (4),

$$x = \frac{1 - 2m^2}{1 - m^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and, since $y = mx$,
$$y = \frac{m - 2m^3}{1 - m^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Regarding m as a variable, (5) and (6) express the coordinates of any point of the curve.

Putting $m = 0$, the line $y = mx$ coincides with the axis of x, and (5) gives $x = 1$, showing that P coincides with A.

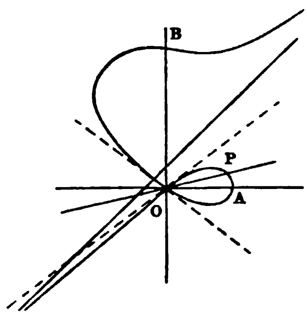


FIG. 20.

As m increases from zero x decreases, and becomes zero when $m = \sqrt{\frac{1}{2}}$, the point P describing the upper part of the loop AO. The position of the line $y = mx$, corresponding to $m = \sqrt{\frac{1}{2}}$, is indicated by the dotted line in Fig. 20. This line is a tangent to the curve at the origin.

As m increases from this value to unity, x becomes negative and passes to infinity, when OP makes an angle of $45°$ with the axis of x; P describing the infinite branch in the third quadrant.

As m passes the value unity, x again changes sign, and decreases; the point P describing the infinite branch in the first quadrant, passing through the point $(1, 2)$ when $m = 2$, and arriving at the point B $(0, 2)$ when $m = \infty$.

As the line OP passes into the second quadrant, m becomes negative and decreases numerically; x also becomes negative, and returns to zero when $m = -\sqrt{\frac{1}{2}}$; P describing the branch BO.

Finally, as m passes through this value and returns to zero, x becomes positive and increases to unity; P describing the branch OA in the fourth quadrant, and returning to the point whence it started.

If it be desired to construct the curve with accuracy, as many points as we choose may be determined by assigning values to m, and computing x and y by equations (5) and (6).

The method of determining the asymptote drawn in Fig. 20 is explained in Art. 228.

The Direction of the Curve.

213. From equation (1) we derive

$$\frac{dy}{dx} = \frac{3x^2 - 2x}{3y^2 - 4y} = \tan\phi. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

This expression, involving both x and y, may be employed to determine the direction of the curve at any point whose coordinates have been determined; for example, the inclination of the curve to the axis of x at the point $(1, 2)$ is $\tan^{-1}\frac{1}{4}$.

The derivative $\frac{dy}{dx}$, which takes the form $\frac{0}{0}$ at the origin, gives, when évaluated by the method of Art. 117, two values of $\tan\phi$ which are identical with the two values of m for which the point P passed through the origin. It is to be remarked that the expression for the derivative $\frac{dy}{dx}$ must necessarily take the indeterminate form at a point through which two or more branches pass, otherwise $\tan\phi$ would have but a single value.

214. The numerator in the expression [equation (7)] for $\tan\phi$ vanishes for $x = 0$ and for $x = \frac{2}{3}$; hence $\tan\phi = 0$ for these values of x, except when the denominator vanishes at the same time. Thus, corresponding to $x = 0$, we have a maximum ordinate at B, while at the origin both terms of the fraction vanish. Since the cubic equation which results from the substitution of $x = \frac{2}{3}$ in equation (1) has three real roots, neither of which reduces the denominator of the expression for $\tan\phi$ to zero, there are three points on the line $x = \frac{2}{3}$ at which $\tan\phi$ vanishes.

In like manner, the denominator vanishes when $y = 0$ and when $y = \frac{4}{3}$. Accordingly, corresponding to $y = 0$ there is a maximum abscissa at the point $(1, 0)$. Putting $y = \frac{4}{3}$ in equation (1), we obtain a cubic equation which has two imaginary

roots, and one real root equal to -0.81 (nearly); there is therefore a minimum abscissa corresponding to $y = \frac{4}{3}$, whose value is -0.81 approximately.

215. *Example* 6. $x^4 - 3axy^2 + 2ay^3 = 0.$ . . . . . (1)

Putting $y = mx$ as in Art. 211, we have

$$x^4 - (3am^2 - 2am^3)x^3 = 0. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This equation has three zero roots; hence, whatever be the value of m, the line $y = mx$ is said to cut the curve in *three coincident points* at the origin.

Rejecting the factor x^3 from equation (2), we have

$$x = 3am^2 - 2am^3 \quad . \quad . \quad . \quad . \quad (3)$$

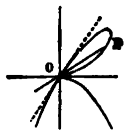


FIG. 21.

for the abscissa of P, the only point distinct from the origin in which the line $y = mx$ cuts the curve; and for the corresponding ordinate we have

$$y = 3am^3 - 2am^4. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

When $m = 0$, equation (3) gives $x = 0$, P being at the origin.

As m increases from zero, x becomes positive, but returns to zero when $m = \frac{3}{2}$, while P describes the loop in the first quadrant.

As m increases from $\frac{3}{2}$ to ∞, x passes from 0 to $-\infty$, while P describes the infinite branch in the third quadrant.

As m changes sign and passes from $-\infty$ to 0, x becomes positive and y negative, and their numerical values decrease to zero, while P describes the infinite branch in the fourth quadrant and returns to the origin.

Maxima Values of the Coordinates Expressed in Terms of an Auxiliary Variable.

216. In finding maxima and minima ordinates and abscissas, it is often convenient to treat y and x as functions of m. Thus, from (3), we obtain

$$\frac{dx}{dm} = 6am - 6am^2;$$

putting this derivative equal to zero, we derive

$$m = 0, \quad \text{and} \quad m = 1.$$

$m = 1$ gives the point (a, a) at which there is a maximum abscissa.

In like manner, from (4), we derive

$$\frac{dy}{dm} = 9am^2 - 8am^3;$$

putting this derivative equal to zero, we obtain

$$m = 0, \quad \text{and} \quad m = \tfrac{9}{8}.$$

By substituting the value $m = \frac{9}{8}$ in equations (3) and (4), we obtain, for the coordinates of a point of the curve which has a maximum ordinate,

$$x = 0.95a \quad \text{and} \quad y = 1.07a.$$

The abscissa corresponding to $m = 0$ is a minimum; but the ordinate is neither a maximum nor a minimum, as may be seen by examining the motion of P as the revolving line OP passes through the position OX, Fig. 21.

Tangents and Points of Inflexion.

217. If we suppose a straight line to pass through a given point of a curve, and to rotate about this point, then, as the line approaches the position of the tangent through the given point, it will have a second intersection with the curve, and this second intersection will coincide with the given point when the rotating line reaches the position of the tangent. For this reason a tangent line is said to meet the curve in *two coincident points* at the point of contact. If however the given

point on the curve is a point of inflexion, the rotating line will evidently have two other intersections, one on each side of this point, and the three intersections will finally come into coincidence. Hence a tangent at a point of inflexion is said to meet the curve in *three coincident points.*

218. When the point of contact is at the origin, the value of $\tan\phi$ at that point, is proved in Art. 118 to be the same as that of $\dfrac{y}{x}$ in the equation resulting from placing equal to zero the terms of lowest degree in the equation of the curve. Thus, in the case of the curve traced in Art. 202, the equation may be written in the form

$$x^2y - 2axy + a^2(y - x) = 0; \quad . \quad . \quad . \quad . \quad (1)$$

the above method gives

$$y - x = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

whence at the origin $\qquad \tan\phi = 1.$

But since equation (2) may be regarded as the equation of a line in the form $y = mx$, in which m denotes the value of $\tan\phi$ at the origin, (2) is itself *the equation of the tangent at the origin.* If now we combine equations (1) and (2) so as to eliminate y, the terms of the first degree will vanish, and the result will contain x^2 as a factor; whence we have the double root $x = 0$, indicating the two coincident points at which the tangent meets the curve.

219. If the given equation contains no terms of the second degree, or if these terms are *divisible by those of the first degree,* the tangent at the origin will meet the curve in more than two coincident points. Thus, if the equation of the curve be

$$x^3 + xy^2 + x^2 - y^2 + x - y = 0,$$

the tangent at the origin,

$$y = x,$$

will cut the curve in *three coincident points*, since the equation obtained by eliminating y from the given equation will contain the triple root $x^3 = 0$. Hence this curve has a point of inflexion at the origin.

220. In like manner if, while the equation of the curve contains terms of the first degree, but no absolute term, the lowest terms which do not vanish with the terms of the first degree are of a degree higher than the third, the tangent at the origin is said to meet the curve in more than three coincident points. Since however there is in this case but a single branch of the curve at the origin, it is impossible for more than two real points to come into coincidence with the given point, hence we infer that some of the intersections of the rotating line with the curve are imaginary; and, since imaginary roots occur in pairs, we have a point of inflexion when the number of coincident points is *odd* but not when that number is *even*.

Nodes and Cusps at the Origin.

221. When the equation of a curve contains no terms lower in degree than the second, it may be shown, as in Art. 211, that *every* straight line passing through the origin meets the curve in *two coincident points*. The curve is in this case said to have a *node* at the origin.

Since a tangent at an ordinary point of a curve meets it in two coincident points, the tangent to either branch of a curve at a node meets the curve in *three coincident points*.

222. The values of tan ϕ at a node are found by the process employed in Art. 218. In the case of the curve

$$x^3 - y^3 - x^2 + 2y^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

constructed in Fig. 20, Art. 212, the equation thus obtained is

$$x^2 - 2y^2 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The two values of $\frac{y}{x}$ deducible from this equation are identical with those of $\tan\phi$ at the origin; hence, regarding x and y as variables, this equation is satisfied by the coordinates of any point on either of the tangents at the origin; in other words, equation (2) is equivalent to the equations of *both tangents at the origin.* Resolving equation (2) into factors, we have

$$x + y\sqrt{2} = 0 \quad \text{and} \quad x - y\sqrt{2} = 0,$$

the separate equations of the two tangents at the origin.

The above reasoning is obviously general; hence we infer that whenever the curve passes through the origin, the equations of the tangents at that point may be obtained *by placing equal to zero the terms of lowest degree in the equation of the curve.*

223. When, as in the above example, the values of $\frac{y}{x}$ are found to be real, we infer that the curve has two branches passing through the origin which in this case is called a *crunode.*

When on the other hand the values of $\frac{y}{x}$ are imaginary, the curve can have no real branches passing through the origin, and the curve is said to have a *conjugate point* or *acnode.*

Finally, if the two tangents are found to be real and coincident, there will usually be two real branches of the curve having a common tangent at the origin and terminating at that point, which is in this case called a *cusp.* When the branches lie on opposite sides of the tangent the cusp is called a *ceratoid* cusp*, and when they lie on the same side it is called a *ramphoid† cusp.*

* From κέρας, a horn. † From ῥάμφος, a beak.

224. It is to be remarked that in some cases when the given equation seems to indicate a cusp at the origin, this point will nevertheless be found to be an isolated point, y being imaginary for values of x near zero. For example, the equation

$$x^6 - a^2x^4 - a^4y^2 = 0$$

indicates the existence of two tangents at the origin coincident with the axis of x; but, on solving for y, we obtain

$$y = \pm \frac{x^2}{a^2} \sqrt{(x^2 - a^2)},$$

whence it is obvious that y is imaginary for all values of x numerically less than a, except zero, the origin is therefore an isolated point.

Multiple Points.

225. When the equation of a curve contains no term lower in degree than the third, the curve is said to have a *triple point* at the origin. In general, when an equation contains no terms lower in degree than the mth, the curve is said to have a *multiple point of the* m*th order* at the origin, and the equation which results from putting equal to zero the terms of lowest degree is the equation of the m tangents at the origin. Thus, resuming example 6, Art. 215, the equation of the curve is

$$x^4 - 3axy^2 + 2ay^3 = 0,$$

whence the equation of the tangents at the origin is

$$3xy^2 - 2y^3 = 0;$$

therefore the tangents are

$$y = 0, \quad y = 0, \quad \text{and} \quad 3x - 2y = 0.$$

The cusp in Fig. 21 corresponds to the double root $y = 0$, the axis of x being the common tangent to the two branches; the other tangent is indicated by the dotted line.

226. A triple point is regarded as equivalent to *three* nodes; for suppose three branches of the curve to pass very near to the same point, as in Fig. 22, these branches will intersect in three nodes. If, now, one of these branches be so moved as to cause the three nodes to coincide, these points may be regarded as uniting to form a triple point.

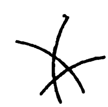


Fig. 22.

In like manner, a multiple point of the order m is equivalent to $\frac{1}{2}m\,(m-1)$ nodes; this being, by the algebraic theory of combinations, the number of intersections that ultimately coincide.

The triple point in Fig. 21 is equivalent to two nodes and a cusp.

The number of *real* branches which exist at the origin may, however, be less than m, since one or more pairs of the tangents may be imaginary (see Art. 223); and, also, because one or more pairs of the branches corresponding to *equal* values of m may be imaginary (see Art. 224). Thus a point which fulfils the condition requisite for a triple point may have but a single branch of the curve passing through it, and may consequently present to the eye nothing to distinguish it from an ordinary point of the curve.

Examples XXVII.

1. Trace the curve $y^3 - x^3 - y + 4x = 0$.

Since the equation contains terms of odd degrees only, if a point (x, y) satisfies the equation, the point $(-x, -y)$ will likewise satisfy it; hence the curve is symmetrical in opposite quadrants. It follows that the asymptote is symmetrically situated, and must therefore pass through the origin. The intersections of the curve with the axes and with the lines $y = \pm 1$, and $x = \pm 2$ are easily obtained; also the

lines on which the points having maxima and minima ordinates are situated.

2. Trace the curve $y^4 - 16x^4 + x^2 - 4xy = 0$.
Find the points at which the curve cuts the axis of x and its direction at these points.

3. Trace the curve $2y^5 - 5xy^2 + x^5 = 0$.
Find the maximum ordinate and the maximum abscissa by means of the derivative expressed in terms of x and y. The former occurs at the point $(\sqrt[5]{2}, \sqrt[5]{4})$, and the latter at $(\sqrt[10]{27}, \sqrt[10]{3})$.

4. Trace the curve $x^5 - 5x^2y - 3xy^2 + y^5 = 0$.
Find the asymptote, and the tangents at the origin. Determine points on the loops by making $m = 1$, and find the direction of the curve at these points.

5. Trace the curve $x^4 - axy^2 + y^4 = 0$.
The tangents to the loop in the first quadrant are parallel to the axes at the points $(\frac{1}{2}a, \frac{1}{2}a)$ and $(\frac{1}{4}\sqrt{3}\,.\,a, \frac{1}{4}\sqrt[4]{27}\,.\,a)$.

6. Trace the curve $x^4 - ax^2y + by^2 = 0$.
Find the points at which the curve is parallel to the coordinate axes.

7. Trace the curve $x^4 + x^2y^2 + y^4 = ax(x^2 - y^2)$.
The minimum abscissa is $x = (1 - \frac{2}{3}\sqrt{3})\,a$; the maxima and minima ordinates are at the points determined by $x = (\frac{1}{4} \pm \frac{1}{12}\sqrt{21})\,a$ and $y = \pm\frac{1}{6}\sqrt{3}\,.\,a$.

8. Trace the curve $x^4 - y^4 - a^2xy = 0$.
$y = x$ and $y = -x$ are asymptotes.

9. Trace the curve $x^4 - 2a^2x^2 - 2ay^3 + 3a^2y^2 = 0$.
The value of x in terms of m may be put in the form

$$x = am^3 \pm a(m^2 - 1)\sqrt{(m^2 + 2)}.$$

The curve has nodes at the points determined by $m = \pm 1$. Determine the direction of the curve at these points, and determine also the

points at which the curve is parallel to the axes, by means of the derivative in terms of x and y.

10. Trace the curve $y^4 + x^4 - 8ay^3 + 6ax^2y = 0$.

The curve is symmetrical with reference to the axis of y and consists of three loops. The minimum ordinates correspond to $m = \pm \frac{1}{3}\sqrt{3}$.

11. Trace the curve $(x^2 + y^2)^2 = 4x^2 + y^2$.

The origin is a conjugate point. To find the maximum ordinates employ the expression for y^2 in terms of m. $m = \infty$ gives the points $(0, \pm 1)$; and $m = \pm\sqrt{2}$ gives the points $(\pm \frac{1}{3}\sqrt{6}, \pm \frac{2}{3}\sqrt{3})$.

12. Trace the curve $(x^2 + y^2)^3 - 4a^2x^2y^2 = 0$.

The curve consists of four loops. A maximum ordinate occurs at $(\frac{2}{9}a\sqrt{6}, \frac{4}{9}a\sqrt{3})$.

13. Trace the curve $y^4 - 96a^2y^2 + 100a^2x^2 - x^4 = 0$.

Find the coordinates of the points at which the abscissa has maxima and minima values. In constructing the curve, determine the points corresponding to $m = \frac{1}{2}$ and $m = 2$. This curve is known as "*la courbe du diable.*"

14. Trace the curve $a^4(x + y)^2 = (a^2 - x^2)^3$.

The curve has two cusps at both of which the line $x + y = 0$ is tangent to the curve. Find the direction of the curve at the points at which it cuts the axis of y.

XXVIII.

Points at Infinity.

227. When the generating point of a curve recedes to infinity, the value assumed by the ratio $\frac{y}{x}$ may be determined by the following method.

Let the equation of the curve be assumed in the general form

$$Ax^n + Bx^{n-1}y \cdots + A'x^{n-1} + \cdots + A''x^{n-2} \cdots = 0, \quad . \quad . \quad . \quad (1)$$

in which the first group of terms is of the nth or highest degree, the second of the $(n-1)$th degree, etc. Dividing the equation by x^n we have

$$A + B\frac{y}{x} \cdots + \frac{1}{x}\left[A' + B'\frac{y}{x} \cdots\right] \cdots = 0.$$

Now when x and y are simultaneously infinite, assuming that $\frac{y}{x}$ has a finite value, this equation reduces to

$$A + B\frac{y}{x} + \cdots = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The equation

$$Ax^n + Bx^{n-1}y + \cdots = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

of which the first member consists of the group of terms of highest degree in equation (1), determines values of $\frac{y}{x}$ identical with those determined by equation (2). If the value of $\frac{y}{x}$ is infinite when x and y are infinite, the reciprocal ratio $\frac{x}{y}$ will have a finite value; therefore by dividing by y^n it may be shown that in this case also equation (3) gives the required values of the ratio of x and y.

In discussing the general equation, we shall frequently use the abridged form

$$P_n + P_{n-1} + \cdots + P_0 = 0,$$

in which P_n indicates the sum of the terms of the nth degree, P_{n-1}, the sum of those of $(n-1)$th degree, and so on. Equation (3) becomes, when this notation is adopted,

$$P_n = 0.$$

Since this equation is of the nth degree it determines n values of the ratio of y to x, and each of these values is said to determine *a point in which the line at infinity cuts the curve.*

Asymptotes.

228. Applying the above method to the curve traced in Art. 212, of which the equation is

$$x^3 - y^3 - x^2 + 2y^2 = 0, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

we obtain, for the equation $P_n = 0$ which determines the points at infinity,

$$x^3 - y^3 = 0,$$

or $\quad (x-y)(x^2 + xy + y^2) = 0.$

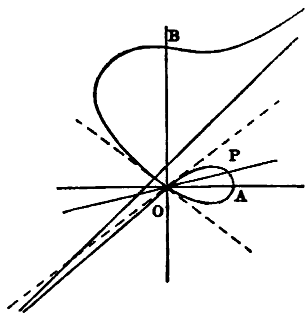


FIG. 20.

The second factor gives imaginary values of the ratio $\frac{y}{x}$; hence the curve has but one *real* point at infinity; namely, that for which

$$\left.\frac{y}{x}\right]_\infty = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

To determine the asymptote the equation of the curve may be put in the form

$$y - x = \frac{2y^2 - x^2}{x^2 + xy + y^2} = \frac{2\frac{y^2}{x^2} - 1}{1 + \frac{y}{x} + \frac{y^2}{x^2}}. \quad . \quad . \quad . \quad (3)$$

If, in this equation, we suppose x and y to become infinite, and substitute the value of $\left.\frac{y}{x}\right]_\infty$ from equation (2), we obtain $\frac{1}{3}$ for the limiting value of the second member; hence, when the generating point recedes to infinity, the value of the first member, $y - x$, approaches indefinitely to the fixed quantity $\frac{1}{3}$. If therefore we put

$$y - x = \tfrac{1}{3}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

we have the equation of a straight line to which the point approaches indefinitely; in other words, *this line is an asymptote.*

229. If the given equation be of the nth degree, terms lower in degree than $n - 1$ will not in general affect the position of the asymptote. Thus, if we have

$$x^3 - xy^2 + ay^2 - a^2y = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

the points at infinity are determined by

$$x(x + y)(x - y) = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

to find the asymptote corresponding to the factor $x - y$, we put the equation in the form

$$x - y = \frac{a^2y - ay^2}{x(x + y)} = \frac{\frac{a^2y}{x^2} - a\frac{y^2}{x^2}}{1 + \frac{y}{x}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Evaluating this expression when $x = y = \infty$, we have for the equation of the asymptote

$$x - y = -\tfrac{1}{3}a. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The term a^2y, which is lower in degree than x^3, does not affect the above result, since the corresponding term in the last member of equation (3) disappears when x is infinite, and, in general, it is evident from the process employed that, *when the factor which determines the asymptote occurs but once in the expression P_n, the position of the asymptote is dependent only upon the terms $P_n + P_{n-1}$.*

The Points at which an Asymptote cuts the Curve.

230. *A straight line parallel to an asymptote is said to meet the curve at infinity;* since it is evident that when a straight line cutting a branch which has an asymptote becomes parallel to the asymptote, one of its intersections with the curve must disappear by passing to infinity. Accordingly, if we eliminate one of the variables by combining the equation of a curve of the nth degree with the equation of a line parallel to an asymptote, the degree of the resulting equation will be denoted by $n - 1$. Now, if the line after becoming parallel to the asymptote be moved into coincidence with it, another intersection will recede to an infinite distance. Hence the asymptote is said *to meet the curve in two coincident points at infinity,* or in other words to be *a tangent at infinity;* in fact, it occupies the limiting position of a tangent line, when the point of contact with the curve passes to infinity. The degree of the equation which results from the elimination of one of the variables between the equation of the curve and that of the asymptote will accordingly be denoted by $n - 2$; and this is the number of finite points in which the asymptote meets the curve.

231. This circumstance facilitates the determination of the points in which the asymptote intersects the curve. Thus, in the case of the example discussed in Art. 229, by combining equations (1) and (4) to eliminate x we obtain

$$-\tfrac{1}{4}a^2y - \tfrac{1}{8}a^3 = 0;$$

hence $$y = -\tfrac{1}{2}a,$$

and from (4) $$x = -a.$$

These are the coordinates of the single point in which the asymptote cuts the curve.

In a similar manner it may be shown that the asymptote corresponding to the factor $x + y$ of equation (2) is

$$y = -x - \tfrac{1}{2}a,$$

and that this line cuts the curve at the point $\left(-\dfrac{3a}{7}, -\dfrac{a}{14}\right)$.

Asymptotes Parallel to the Coordinate Axes.

232. Whenever x or y is a factor of the terms P_n, the corresponding asymptote is of course parallel to one of the coordinate axes ($x = 0$ or $y = 0$). In such cases, the position of the asymptote is most readily found by the process employed below.

In the case of the curve discussed in Art. 229, since x is a factor of equation (2), the corresponding asymptote is parallel to the axis of y ($x = 0$). Arranging the equation of the curve with reference to powers of y, we have

$$(x - a)y^2 + a^2y - x^3 = 0. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

For each value of x substituted in this equation we have two

values of y; hence a line parallel to the axis of y generally cuts the curve in two points. If however the value of x substituted causes the coefficient of y^2 to vanish, the corresponding line will have but one point of intersection with the curve, hence this line must be an asymptote.

In the above example the coefficient of y^2 is $x - a$; the equation of the required asymptote is, therefore,

$$x - a = 0.$$

By making $x = a$ in equation (5), we obtain the coordinates (a, a) of the point at which this asymptote cuts the curve.

The above process shows that, *whenever the expression P_n is divisible by one of the variables, the equation of an asymptote is found by placing equal to zero the coefficient of the highest power of the other variable which appears in the given equation.*

Parabolic Branches.

233. In the case of the curve constructed in Art. 215; viz.,

$$x^4 - 3axy^2 + 2ay^3 = 0,$$

the points at infinity are determined by the equation

$$x^4 = 0,$$

therefore the only points at infinity are in the direction of the axis of y. There is, however, no asymptote parallel to this axis, since the coefficient of the highest power of y $(2a)$ is independent of x, and therefore does not vanish for any value of x. Hence the infinite branches of this curve have no asymptotes.

An infinite branch of this kind is called a *parabolic branch*,

because the parabolas furnish the most familiar instances of infinite branches without asymptotes.

234. *A curve can have a parabolic branch only when the equation $P_n = 0$ has two or more equal roots.* For, if we apply the method of determining an asymptote given in Art. 228, when the branch is parabolic the fraction in the second member must evidently take an infinite value when $x = \infty$. Now the denominator of this fraction is of the $(n - 1)$th degree, and the numerator cannot be of a higher degree; hence, if each term of the fraction be divided by x^{n-1}, the numerator cannot become infinite when $x = \infty$. It follows that the value of the fraction can become infinite only when the substitution of the value of $\left.\frac{y}{x}\right]_\infty$ causes the denominator to vanish: but, in order that this may be the case, it is necessary that the factor of P_n, which determines the value of $\frac{y}{x}$ employed, shall also be a factor of the denominator; in other words this factor must occur at least twice in P_n.

On the other hand, if P_n contains the square of a factor *not contained in* P_{n-1}, the value of the fraction must be infinite for infinite values of x, and the branches corresponding to this double root must be parabolic. Thus, if the equation of the curve is

$$(x - 2y)^2 (x + y) - a (x^2 + y^2) + \cdots = 0,$$

the branches corresponding to $(x - 2y)^2 = 0$ are parabolic, as may be seen by writing the equation in the form

$$x - 2y = \frac{a (x^2 + y^2) + \cdots}{(x + y)(x - 2y)},$$

and putting $$x = 2y = \infty.$$

A double root of the equation $P_n = 0$ is not generally to be interpreted as indicating the existence of a double point at infinity; it only indicates that *the line at infinity* meets the curve in two coincident points. In the case of a parabolic branch, this line is regarded as a tangent to the curve; for it is evident that as the point of contact of a tangent line recedes indefinitely on a parabolic branch, the tangent recedes indefinitely, instead of approaching a limiting position as in the case of a branch having an asymptote.

Parallel Asymptotes.

235. We have now to consider the case in which the square of a factor appears in P_n, and the first power of the same factor in P_{n-1}. For example, let the given equation be

$$2x(x-y)^2 - 3a(x^2-y^2) + 4a^2y - 7a^3 = 0. \quad . \quad . \quad (1)$$

This equation may be put in the form

$$(x-y)^2 - \frac{3a(x+y)}{2x}(x-y) + \frac{2a^2y}{x} - \frac{7a^3}{2x} = 0. \quad . \quad . \quad (2)$$

Equation (2) may be regarded as a quadratic for determining the value of $x - y$, when x is infinite.

Putting $y = x = \infty$ in the coefficients, the equation reduces to

$$(x-y)^2 - 3a(x-y) + 2a^2 = 0;$$

whence we have

$$x - y = 2a \quad \text{and} \quad x - y = a,$$

the equations of two parallel asymptotes. It will be observed

that in finding the equations of a pair of parallel asymptotes it is necessary to employ the terms included in the expression $P + P_{n-1} + P_{n-2}$.

236. When a curve has a pair of parallel asymptotes, it may be shown, by the method employed in Art. 230, that every straight line parallel to them meets the curve in two coincident points at infinity. Accordingly, if we eliminate one of the variables by combining the equation of such a line with the equation of the given curve, the degree of the resulting equation will be denoted by $n-2$. This circumstance facilitates the determination of the intersections of the line with the curve, and the expressions for the coordinates of the point or points of intersection may frequently be used in tracing the curve, as in the following example.

Example 7. $x(y+x)^2 + a^2y = 0.$ . . . . (1)

The method employed in the preceding article gives the equations of two parallel asymptotes; viz.,

$$y + x = a \quad \text{and} \quad y + x = -a.$$

The curve has evidently a third asymptote

$$x = 0.$$

The point in which the straight line

$$y + x = \beta$$

meets the curve is determined by the equations,

$$x = \frac{a^2\beta}{a^2 - \beta^2} \quad \text{and} \quad y = \frac{-\beta^3}{a^2 - \beta^2}:$$

if now we make β a variable, this point will describe the curve.

Beginning with $\beta = 0$, we have $x = 0$ and $y = 0$; P is therefore at the origin.

As β increases from zero, x is positive and y is negative; each numerically increases until $\beta = a$, when both are infinite, P describing the infinite branch in the fourth quadrant tangent to the axis of x at the origin. See Fig. 23.

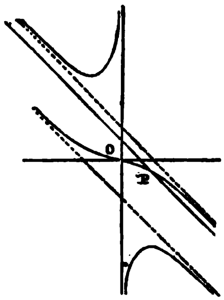


FIG. 23.

As β increases from the value a, x is negative and numerically decreases to zero, the branch described by P approaching the axis of y as an asymptote.

Again, since the given equation is unaltered by changing x to $-x$ and y to $-y$, it is evident that the curve must be symmetrical in opposite quadrants.

The axis of x is tangent to the curve at a point of inflexion. See Art. 219.

To find the maximum and the minimum ordinate, we derive from the value of y in terms of β,

$$\frac{dy}{d\beta} = \frac{\beta^2(\beta^2 - 3a^2)}{(a^2 - \beta^2)^2};$$

this derivative changes sign only when

$$\beta^2 - 3a^2 = 0,$$

whence $$\beta = \pm a\sqrt{3}.$$

These values substituted in the expressions for x and y determine the points

$$(-\tfrac{1}{2}a\sqrt{3}, \tfrac{3}{2}a\sqrt{3}) \quad \text{and} \quad (\tfrac{1}{2}a\sqrt{3}, -\tfrac{3}{2}a\sqrt{3});$$

the former having a minimum and the latter a maximum ordinate.

Nodes at Infinity.

237. A curve having two parallel asymptotes, like the one constructed in the preceding article, is said to have *a node at infinity*, because every straight line parallel to the asymptotes, that is, every line passing through the same point at infinity, meets the curve in two coincident points; whereas, in the case of a parabolic branch (which may likewise occur when the equation for determining the points at infinity has a pair of equal roots) it is only *the line at infinity* that meets the curve in two coincident points. See Art. 234.

Parallel asymptotes correspond to the two tangents at an ordinary node; when these asymptotes are found to be imaginary the curve is said to have an *acnode at infinity*. Thus, if we change the sign of the term a^2y, in equation (1) Art. 236, we have

$$x(y+x)^2 - a^2y = 0.$$

The asymptotes of this curve are

$$y + x = \pm a\sqrt{(-1)};$$

since these lines are imaginary, the point at infinity is an acnode.

This curve may however be traced by the method explained in the preceding article, since a straight line whose equation is of the form

$$y + x = \beta$$

cuts the curve in a single point.

The curve consists of a single branch having for an asymptote the axis of y, and having a maximum abscissa at the point $(\frac{1}{2}a, \frac{1}{2}a)$.

238. When $P_n = 0$ has three equal roots, the curve may have three asymptotes. This will be the case when P_{n-1} contains the square, and P_{n-2} the first power of the factor whose cube appears in P_n. Thus the equation

$$x(2x - y)^3 - a(x + y)(2x - y)^2 + a^2 y = 0$$

fulfils the required conditions with respect to the factor $2x - y$, since, in this case,

$$P_{n-2} = 0 = 0(2x - y).$$

Making $y = 2x$ in the coefficients of the powers of $2x - y$ and dividing by x, we have, for determining the value of $2x - y$ when x is infinite, the cubic equation

$$(2x - y)^3 - 3a(2x - y)^2 + 2a^3 = 0;$$

whence we derive

$$2x - y = a \quad \text{and} \quad 2x - y = a(1 \pm \sqrt{3}),$$

the equations of three real parallel asymptotes. This curve is accordingly said to have *a triple point at infinity.*

239. When either of the conditions given in the preceding article is not fulfilled, the curve will have parabolic branches only, *except when P_{n-1} contains the first power of the factor whose cube appears in P_n.* In the latter case a single asymptote may be found by the method employed in the following example.

Let the equation of the curve be

$$(x - y)^3 + a(x - y)(3x - 4y) + a^2(3x - 2y) + 2a^3 = 0.$$

In this case, after putting $y = x$ in the coefficients of the

powers of $(x-y)$, it is necessary to divide by x to obtain terms which have a finite value when x is made infinite; this causes the first term to vanish, and gives the simple equation

$$-a(x-y)+a^2=0,$$

hence $$x-y-a=0$$

is the equation of an asymptote.

The remaining infinite branches of the curve are parabolic, hence the line at infinity touches the curve at the point at which the asymptote also touches it; the curve is therefore said to have *a double point at infinity at which the line at infinity is one of the tangents.*

This curve is a *trident;* since the characteristic singularity of a trident is a node of the kind described above. Compare Art. 209. The curve may be traced by the method employed in Art. 236.

240. If the equation employed in determining parallel asymptotes has two equal roots, the curve has two coincident asymptotes, and, in general, two real branches approaching the same part of the common asymptote on opposite sides and forming a *cusp at infinity.* See Fig. 17, Art. 202.

The corresponding branches may, however, in a case of this kind be imaginary, as in the following example:

$$x^4y^2+a^4x^2-a^6=0.$$

Each coordinate axis presents a case of coincident asymptotes. Solving the equation, we obtain

$$y=\pm\frac{a^2}{x^2}\sqrt{(a^2-x^2)};$$

$y = 0$ when $x = \pm a$; as x numerically decreases, both values of y increase indefinitely, giving the branches represented by the full lines in Fig. 24. When x numerically exceeds a, y is imaginary; hence there are no real branches approaching the axis of x. If however we change the sign of the term containing y^2 in the given equation, we shall have the curve

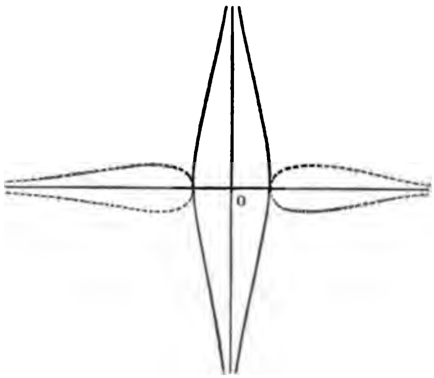


FIG. 24.

$$y = \pm \frac{a^2}{x^2} \sqrt{(x^2 - a^2)},$$

represented by the dotted branches in the figure; the real branches of this curve being those to which the axis of x is an asymptote.

Examples XXVIII.

Find the asymptotes of the following curves.

1. $(x + a)y^2 = (y + b)x^2$. $\quad x = -a, y = -b, y = x + b - a$.

2. $x^3 - 4xy^2 - 3x^2 + 12xy - 12y^2 + 8x + 2y + 4 = 0$.

$$x = -3,\ x = 2y,\ x + 2y = 6.$$

3. $(y - 2x)(y^2 - x^2) - a(y - x)^2 + 4a^2(x + y) - a^3 = 0$.

$$y = x,\ y + x = \tfrac{3}{4}a,\ y = 2x + \tfrac{1}{3}a.$$

4. $x^2y^2 + ax(x + y)^2 - 2a^2y^2 - a^4 = 0$. $\quad x = -2a, x = a$.

5. $x^4y^4 - (x^2 - y^2)^2 + y^2 - 1 = 0.$ $\qquad x = \pm 1, y = \pm 1.$

6. $(y - x)(x^2 - a^2) = a^3.$ $\qquad y = x, x = \pm a.$

7. $x^7 - x^3y^4 + a^4y^3 - ax^2y^4 = 0.$

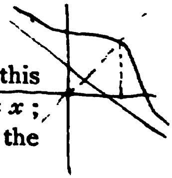
$x = 0, x = -a, x + y = \frac{1}{4}a, x - y = \frac{1}{4}a.$

8. $x^2(x - y)^2 - a^2(x^2 + y^2) = 0.$ $\qquad x = \pm a, y = x \pm a\sqrt{2}.$

9. $(x^2 - y^2)^2 - 4y^2 + y = 0.$ $\qquad x - y = \pm 1, x + y = \pm 1.$

√ 10. Trace the curve $x^3 + y^3 - x^2 - y^2 = 0$.

The origin is an acnode. Since x and y are interchangeable in this equation, the curve is symmetrical with reference to the line $y = x$; find the point at which the curve cuts this line, and show that the asymptote $x + y = \frac{2}{3}$ does not cut the curve.



11. Trace the curve $x^5 + y^5 - 5ax^3y = 0$.

The asymptote is $y + x = -a$, cutting the curve in the second quadrant. The point $(a\sqrt[5]{162}, a\sqrt[5]{108})$, has a maximum ordinate, and the point $(2a\sqrt[5]{8}, 2a\sqrt[5]{2})$ has a maximum abscissa.

12. Trace the curve $y^4 + 2axy^2 - x^4 = 0$.

Find the two asymptotes, and show that each intersects the curve at points corresponding to $x = \pm \frac{a}{4}\sqrt{2}$.

13. Trace the curve $x^5 - 2a^3xy + y^5 = 0$.

The asymptote passes through the origin; the curve is symmetrical with reference to the line $y = x$, and cuts it at the point (a, a).

14. Trace the curve $x^5 - 5ax^2y^2 + y^5 = 0$.

The equation of the asymptote is $y + x = a$.

15. Trace the curve $x^3 - y^3 + (2y - x)^2 = 0$.

Determine the direction of the curve at the points in which it cuts the coordinate axes, and at the point determined by $m = -1$. The curve has a cusp at the origin.

16. Trace the curve $x^2y^2 - x^3y + x^2 - 4y^2 = 0$.
The four asymptotes are $x = \pm 2, y = x, y = 0$.

17. Trace the curve $x^2y - y^2x + x^2 - 4y^2 = 0$.
Find the asymptotes, the points at which they cut the curve and the direction of the curve at these points.

18. Trace the curve $xy^2 + x^2y + 2x^2 - 3xy - 2y^2 = 0$.
Find the three asymptotes, and the point in which each cuts the curve.

19. Trace the curve $4x^3 - 4xy^2 + ay^2 - 3ax^2 = 0$.
Show that the three asymptotes pass through a common point.

20. Trace the curve $(x + 2y)(x - y)^2 - 6a^2(x + y) = 0$.
The curve is symmetrical in opposite quadrants, and has three asymptotes two of which are parallel.

21. Trace the curve $y(y - x)^2(y + 2x) = 3a^2x^2$.
The curve is symmetrical in opposite quadrants, and has four asymptotes two of which are parallel.

22. Trace the curve $y^4 - y^2x + x^3 - 2x^2y = 0$.
The curve has parabolic branches, and an asymptote that cuts the curve in two points.

23. Trace the curve $x^4 - ax^2y - axy^2 + a^2y^2 = 0$.
The origin is an isolated point (see Art. 224). The curve has parabolic branches, and the line $x = a$ is an asymptote that cuts the curve in a single point. The line $y = mx$ touches the curve when $m = 1$ and when $m = -3$. By solving for y, it may be shown that the ine $x = \frac{3}{4}a$ touches the curve.

24. Trace the curve $x^4 - ax^2y + axy^2 + \frac{1}{4}a^2y^2 = 0$.
The equation of the asymptote is $x = -\frac{1}{4}a$. The curve has parabolic branches and a ramphoid cusp at the origin.

25. Trace the curve $x^4 - \frac{8}{9}ax^2y - axy^2 + a^2y^2 = 0$.

The line $y = mx$ touches the curve when $m = -\frac{1}{2}$, and when $m = -\frac{3}{2}$. Show that both branches of the curve pass through the origin, and find the asymptote and the point at which it cuts the curve.

26. Trace the curve $x^5 - 4ay^4 + 2ax^3y + a^2xy^2 = 0$.

The origin is a triple point at which two of the branches form a ramphoid cusp. The line $y = x$ touches the curve at (a, a). The infinite branches are parabolic.

27. Trace the curve $(x - y)^3 + a(x^2 - y^2) + a^2y = 0$.

This curve is a trident touching the axis of x at the origin. See Art. 239.

XXIX.

Curves Given by Polar Equations.

241. The following examples will illustrate some of the methods employed when the curve is given by means of its polar equation.

Example 8. $r = a \cos\theta \cos 2\theta$. . . . . (1)

When $\theta = 0$, $r = a$, the generating point P therefore starts from A on the initial line. As θ increases, r decreases and becomes zero when $\theta = 45°$, P describing the half-loop in the first quadrant, and arriving at the pole in a direction having an inclination of $45°$ to the initial line. When θ passes $45°$, r becomes negative, and returns to zero again when $\theta = 90°$, P describing the loop in the third quadrant. As θ passes $90°$, r again becomes positive, but returns to zero when $\theta = 135°$, P describing the loop in the second quadrant. As θ varies from $135°$ to $180°$, r again becomes negative, P describing the half-loop in the fourth quadrant, and returning to A.

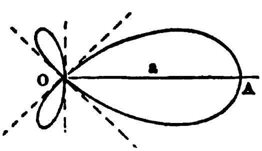


FIG. 25.

In this example if we suppose θ to vary from 180° to 360°, P will again describe the same curve, and, since θ enters the equation of the curve, by means of trigonometrical functions only, it is unnecessary to consider values of θ greater than 360°.

242. Putting equation (1) in the form

$$r = a(2\cos^3\theta - \cos\theta),$$

we derive

$$\frac{dr}{d\theta} = a(-6\cos^2\theta \sin\theta + \sin\theta).$$

To determine the maxima values of r, we place this derivative equal to zero, thus obtaining the roots

$$\sin\theta = 0 \quad \text{and} \quad \cos\theta = \pm\tfrac{1}{6}\sqrt{6};$$

the former gives the point A on the initial line, and the latter gives the values of θ which determine the position of the maxima in the small loops. The corresponding values of r are $\mp\dfrac{a}{9}\sqrt{6}$.

To determine the position of the maximum ordinate, we have from (1)

$$y = r\sin\theta = \tfrac{1}{4}a\sin 4\theta.$$

The maxima values occur when $\sin 4\theta = 1$, and the minima when $\sin 4\theta = -1$; that is, we have maxima when $\theta = \frac{1}{8}\pi$ and when $\theta = \frac{5}{8}\pi$, and minima when $\theta = \frac{3}{8}\pi$ and $\frac{7}{8}\pi$.

243. In the preceding example the substitution of $\theta + \pi$ for θ changes the sign but not the numerical value of r. When this is the case, θ and $\theta + \pi$ evidently give the same point of the curve, and the complete curve is described while θ varies from 0 to π. If however this substitution changes neither the numerical value nor the sign of r, θ and $\theta + \pi$ will give points symmetrically situated with reference to the pole; that is, the curve will be symmetrical in opposite quadrants.

Again if the substitution of $-\theta$ for θ does not change the value of r, θ and $-\theta$ give points symmetrically situated with reference to the initial line, hence in this case the curve is symmetrical to this line; but, if the substitution of $-\theta$ for θ changes the sign of r without changing its numerical value, the curve is symmetrical with reference to a perpendicular to the initial line.

The Determination of Asymptotes by Means of Polar Equations.

244. When r becomes infinite for a particular value of θ the curve has an infinite branch, and, if there be a corresponding asymptote, it may be determined by means of the expression derived below.

Let θ_1 denote a value of θ for which r is infinite, and let OB be drawn through the pole, making this angle with the initial line; then, from the triangle OBP, Fig. 26, we have

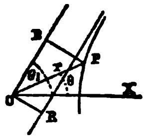


FIG. 26.

$$PB = r \sin(\theta_1 - \theta).$$

Now, if the curve has an asymptote parallel to OB, it is plain that as θ approaches θ_1 the limiting value of PB will be equal to OR, the perpendicular from the pole upon the asymptote. Hence, if the curve has an asymptote in the direction θ_1, the expression

$$OR = [r \sin(\theta_1 - \theta)]_{\theta_1},$$

which takes the form $\infty \cdot 0$, will have a finite value, and this value will determine the distance of the asymptote from the pole. Fig. 26 shows that when the above expression is positive OR is to be laid off in the direction $\theta_1 - 90°$.

If upon evaluation the expression for OR is found to be in-

finite we infer that the infinite branch of the curve is parabolic.

245. *Example* 9. $$r = \frac{a\theta^2}{\theta^2 - 1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

Since r becomes infinite when $\theta = 1$, we proceed to apply the method established in the preceding article for determining the existence of an asymptote. In this case we have

$$[r \sin(\theta_1 - \theta)]_{\theta_1} = \left[\frac{a\theta^2}{\theta + 1} \cdot \frac{\sin(1 - \theta)}{\theta - 1}\right]_1 = -\tfrac{1}{2}a.$$

The angle $\theta = 1$ corresponds to $57° \; 18'$, nearly, and since the expression for the perpendicular on the asymptote is negative its direction is $\theta_1 + 90° = 147° \; 18'$; consequently, the asymptote is laid off as in Fig. 27.

Numerically equal positive and negative values of θ give the same values for r; hence the curve is symmetrical with reference to the initial line.

While θ varies from 0 to 1, r is negative and varies from 0 to ∞, giving the infinite branch in the third quadrant.

As θ passes the value unity, and increases indefinitely, r becomes positive and decreases, approaching indefinitely to the limiting value a, which we obtain from (1) by making θ infinite. Hence the curve describes an infinite number of whorls approaching indefinitely to the circle $r = a$, which is therefore called *an asymptotic circle.*

The points of inflexion in this curve are determined in Art. 325.

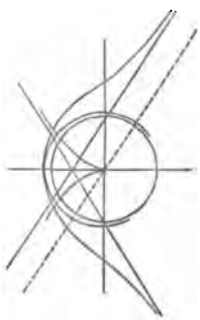


FIG. 27.

Examples XXIX.

1. Trace the curve $r = a \cos^3 \frac{1}{3}\theta$.

Show that, to describe the curve, θ must vary from 0 to 3π; also that the curve is symmetrical to the initial line. Find the values of θ which correspond to the maxima and minima ordinates and abscissas, the initial line being taken as the axis of x.

2. Trace the curve $r = a(2 \sin\theta - 3 \sin^3\theta)$.

Show that the entire curve is described while θ varies from 0 to π, and that the curve is symmetrical with reference to a perpendicular to the initial line.

3. Trace the curve $r = 2 + \sin 3\theta$.

A maximum value of r (equal to 3) occurs at $\theta = 30°$; a minimum (equal to 1) at $\theta = 90°$. The curve is symmetrical with reference to lines inclined at the angles 30°, 90°, and 150° to the initial line.

4. Trace the curve $r = 1 + \sin 5\theta$.

The curve consists of five equal loops.

5. Trace the curve $r^2 = a^2 \sin 3\theta$.

The curve consists of three equal loops.

6. Trace the curve $r \cos\theta = a \cos 2\theta$.

The curve has an asymptote perpendicular to the initial line at the distance a on the left of the pole.

7. Trace the curve $r = 2 + \sin \frac{3}{2}\theta$.

A maximum value of r occurs at $\theta = 60°$, and a minimum at $\theta = 180°$. The curve has three double points, one being on the initial line.

8. Trace the curve $r \cos 2\theta = a$.

The curve is symmetrical with reference to the initial line and with reference to a perpendicular to the initial line. There are four asymptotes.

9. Trace the curve $r \sin 4\theta = a \sin 3\theta$.

The curve is symmetrical to the initial line, and has three asymptotes; the minimum value of r is $\frac{3}{4}a$.

10. Trace the curve $r^2 = a^2 \cos 2\theta$.

The curve is symmetrical with respect to the pole since $r = \pm a \sqrt{(\cos 2\theta)}$: r is imaginary for values of θ between $\frac{1}{4}\pi$ and $\frac{3}{4}\pi$.

11. Trace the curve $r^{\frac{3}{4}} = a^{\frac{3}{4}} \cos \frac{3}{4}\theta$.

The curve consists of three equal loops, r being real for all values of θ.

12. Trace the curve $r = a(\cos\theta + \cos 2\theta)$.

The curve consists of three loops with a cusp at the origin. Determine the two double points by the condition that θ and $\theta + 180°$ shall give the same point of the curve.

13. Trace the curve $r^{\frac{5}{3}} = a^{\frac{5}{3}} \cos \frac{5}{3}\theta$.

The curve consists of five equal loops.

14. Trace the curve $r^2 \sin\theta = a^2 \cos 2\theta$.

The curve consists of two loops and two infinite branches, to which the initial line is a common asymptote.

15. Trace the curve $r^2 \cos\theta = a^2 \sin 3\theta$.

The curve consists of two loops and an infinite branch which has an asymptote perpendicular to the initial line and passing through the pole.

16. Trace the curve $r \sin\theta = a \sin(2\theta + \alpha)$.

Show that θ and $\theta + \pi$ give the same point of the curve. The curve cuts the asymptote at the point $\theta = \frac{1}{2}\pi - \alpha$, $r = a \tan\alpha$.

17. Trace the curve $r = a(\tan\theta - 1)$.

Show that the curve is symmetrical with reference to the pole as a centre, and determine the maximum abscissa.

18. Trace the curve $r = a \dfrac{2\theta}{2\theta - 1}$.

Find the rectilinear and the circular asymptote, and also the point of inflexion.

19. Trace the curve $r = 1 \pm \sqrt{(2 - \operatorname{cosec}\theta)}$.

Show that, to describe the curve, θ must vary from 0 to 2π, and that it consists of three branches, one of which is a closed curve, the other two having a common asymptote.

20. Determine the asymptotes of the curve

$$r \cos m\theta = a \cos n\theta.$$

$$\theta_1 = \frac{(2k+1)\pi}{2m}; \quad OR = (-1)^k \frac{a}{m} \cos \frac{n(2k+1)\pi}{2m}.$$

21. Trace the curve $r(\theta - \pi)^2 = a(\theta^2 - \frac{1}{4}\pi^2)$.

The infinite branches are parabolic; the negative values of r corresponding to values of θ between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ determine a loop; the curve approaches the asymptotic circle both from within and from without. Find the point at which the parabolic branch cuts the asymptotic circle.

22. Trace the curve $r = \dfrac{a\theta}{\theta + \sin\theta}$.

The curve is symmetrical to the initial line; values of θ numerically less than π give a loop enclosing the pole; for greater values of θ each whorl cuts the asymptotic circle at its intersection with the initial line.

23. Trace the curve $r = \dfrac{a(\theta + \cos\theta)}{\theta + \sin\theta}$.

The curve has an asymptote parallel to the initial line; the whorls cut the asymptotic circle at the extremities of a diameter inclined at an angle of 45° to the initial line.

XXX.

Transcendental Curves.

246. The algebraic curves constructed in the preceding sections furnish examples of points of inflexion, nodes, cusps and multiple points. The term *singular point* is used to include these and also two varieties of points which occur only in curves whose equations involve transcendental functions. The following examples of transcendental curves will serve to illustrate these two varieties of singular points; viz., *stop points* or *points d'arrêt* at which a curve terminates abruptly, and *salient points* or *points anguleux* at which two branches of a curve meet without having a common tangent.

Stop Points.

247. *Example* 10. $y = \epsilon^{\frac{1}{x}}$.

Since, by Art. 113, this function has the limiting value infinity when x is positive and approaches zero, and the limiting value zero when x is negative and approaches zero, the right-hand branch has the axis of y for an asymptote, while the left-hand branch stops at the origin.

When x is infinite, $y = 1$; hence the line $y = 1$ is an asymptote to each branch of the curve.

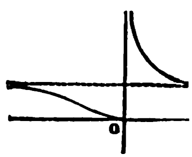


FIG. 28.

Taking the derivative, we have

$$\frac{dy}{dx} = -\frac{\epsilon^{\frac{1}{x}}}{x^2},$$

which, by Art. 114, is zero when x is negative and approaches

zero. Hence the left-hand branch is tangent to the axis of x, and the form of the curve is that represented in Fig. 28.

Taking the second derivative,

$$\frac{d^2y}{dx^2} = \frac{(2x+1)\varepsilon^{\frac{1}{x}}}{x^4},$$

we find a point of inflexion corresponding to $x = -\frac{1}{2}$; the ordinate of this point is $\varepsilon^{-2} = 0.14 \ldots.$

Salient Points.

248. *Example* 11. $\qquad y = \dfrac{x}{1 + \varepsilon^{\frac{1}{x}}}.$

In this case we evidently have $y = 0$, whether x approaches zero from the positive or from the negative side.

The value of $\tan \phi$ at the origin is (see Art. 118) the same as that of $\left.\dfrac{y}{x}\right]_0$, but in this case

$$\left.\frac{y}{x}\right]_0 = \left.\frac{1}{1 + \varepsilon^{\frac{1}{x}}}\right]_0.$$

When x is positive and approaches zero, the value of this expression is zero, but when x is negative and approaches zero its value is unity; hence the right-hand branch touches the axis of x at the origin, while the left-hand branch is inclined at an angle of 45°. See Fig. 29.

249. To determine the infinite branches of this curve we have, from equation (1),

$$\left.\frac{y}{x}\right]_\infty = \frac{1}{2}.$$

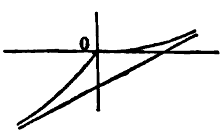


FIG. 29.

To ascertain whether the curve has an asymptote, we put the equation in the form

$$y - \tfrac{1}{2}x = \frac{x}{1 + \varepsilon^{\frac{1}{x}}} - \tfrac{1}{2}x = \frac{x}{2} \cdot \frac{1 - \varepsilon^{\frac{1}{x}}}{1 + \varepsilon^{\frac{1}{x}}},$$

and evaluate for $x = \infty$.

Putting $\frac{1}{x} = z$, we have

$$y - \tfrac{1}{2}x\Big]_\infty = \frac{1}{2(1 + \varepsilon^z)} \cdot \frac{1 - \varepsilon^z}{z}\Big]_0 = -\tfrac{1}{2}.$$

Hence the equation of the asymptote is

$$y = \tfrac{1}{2}x - \tfrac{1}{2}.$$

Branches Pointillées.

250. *Example* 12. $\qquad y = x^x.$

Whence $\qquad \dfrac{dy}{dx} = x^x(1 + \log x);$

since by evaluating we have

$$x^x]_0 = 1,$$

$y = 1$ and $\tan\phi = \infty$ when $x = 0$; hence the curve touches the axis of y at the point $(0, 1)$. It passes through the point $(1, 1)$ at an inclination of 45° to the axis of x, and the ordinate is a minimum at the point $\left[\frac{1}{\varepsilon}, \left(\frac{1}{\varepsilon}\right)^{\frac{1}{\varepsilon}}\right]$.

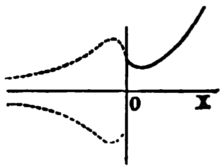


FIG. 30.

251. When x is negative the function x^x is not continuous; it is in fact real only when the value of x can be expressed by an integer or by a fraction having an odd denominator. When

this is the case, the numerical value of each real ordinate is the reciprocal of the ordinate corresponding to a numerically equal positive value of x: hence there is an unlimited number of points situated on branches having the shape indicated by the dotted lines in Fig. 30.

Branches having this character are called *branches pointillées.*

When x is a positive fraction having, when reduced to its lowest terms, an even denominator, the value of x^x is obtained by extracting a root which admits of the ambiguous sign. If the symbol x^x is regarded as expressing at once both values of the function (that is, if for example when $x = \frac{1}{2}$, we write $x^x = \pm \sqrt{\frac{1}{2}}$) the curve $y = x^x$ must be regarded as having also a *branche pointillée* in the fourth quadrant, symmetrical to the continuous branch.

Examples XXX.

1. Trace the curve $y = x \log x$.

Show that there is a stop point at the origin. Find the point at which y is a minimum, and the direction of the curve at the origin, at the point $(1, 0)$, and at the point (ϵ, ϵ).

2. Trace the curve $y^2 = 1 - \cos x$.

The curve has a stop point at the origin; to find the direction of the curve at this point, evaluate $\dfrac{y}{x} = \left(\dfrac{1 - \cos x}{x^2}\right)^{\frac{1}{2}}$ for $x = 0$.

3. Trace the curve $y\left(1 + \epsilon^{\frac{1}{x}}\right) = 1$.

Find the asymptote and the direction of the curve at the two stop points. Show, by transferring the origin to the point $(0, \frac{1}{2})$, that the curve is symmetrical about this point as a centre.

4. Trace the curve $y\left(\epsilon^{\frac{1}{x}} - 1\right) = x\left(\epsilon^{\frac{1}{x}} + 1\right)$.

The curve has a salient point at the origin, and is symmetrical with reference to the axis of y; the branches are parabolic.

5. Trace the curve $y(1 + \varepsilon^{\frac{1}{x}}) = x(1 - \varepsilon^{\frac{1}{x}})$.

The curve has a salient point at the origin, and an asymptote parallel to the axis of x.

6. Trace the curve $y(1 + \varepsilon^{\frac{1}{x}}) = x(1 - 3\varepsilon^{\frac{1}{x}})$.

There is a salient point at the origin. Find the point in which the curve cuts the axis of x, and find also the asymptote.

7. Trace the curve $y = x\varepsilon^{-x}$.

Find the point of inflexion, and the point at which the ordinate has a maximum value.

8. Trace the curve $y = \varepsilon^{\cos x}$, and find the points of inflexion.

9. Trace the curve $y^2\varepsilon^{2x} - x^2 + 1 = 0$.

Find the points at which $\tan\phi$ is zero; also those at which $\tan\phi$ is infinite. The axis of x is an asymptote.

10. Trace the curve $y = (1 - x)^x$.

When $x > 1$, we have *branches pointillées*. Find the inclination of the curve at the points for which $x = -1$, $x = 0$, and $x = 1$.

11. Trace the curve $y = x^{\frac{1}{x}}$.

The curve has a stop point at the origin, and a *branche pointillée* in the second quadrant. Determine the maximum ordinate, the inclination of the curve at the point $(1, 1)$, and also at the origin (the latter by the method employed in Art. 248).

12. Trace the curve $y = \varepsilon^{-x^2}$, and find the points of inflexion.

This curve occurs in the Theory of Least Squares and is known as the Probability Curve.

13. Trace the curve $y = \varepsilon^{\sin^{-1}x}$.

The curve consists of an unlimited number of branches, each containing a point of inflexion. Find the point of inflexion in the branch

corresponding to the primary value of x and the angle at which this branch cuts the axis of y.

14. Trace the curve $y^x = 1 - x$.

The curve has an asymptote, and the portion corresponding to values of x greater than unity consists of *branches pointillées*. Find the inclination of the continuous branch at the points at which it cuts the axes.

15. Trace the curve $y = (\sin x)^x$.

Find the direction of the curve when $x = 0$ and when $x = \pi$. The curve has alternately continuous branches and *branches pointillées*.

CHAPTER IX.

The Equations and Constructions of Certain Higher Plane Curves.

XXXI.

252. In this chapter are given the definitions, equations, and constructions of certain curves, some of which possess an interest chiefly historical, while others, on account of their peculiar properties, are of frequent occurrence in works pertaining to mathematical subjects.*

The Parabola of the n*th Degree.*

253. The term *parabola* is frequently applied to any curve in which one of the coordinates is proportional to the nth power of the other, n being greater than unity. The parabola proper is thus distinguished as the parabola of the second degree.

The general equation of the parabola of the nth degree is usually written in the homogeneous form, (a being positive)

$$a^{n-1}y = x^n.$$

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* A full index to the curves given in this chapter will be found in the table of contents. The cuts have been prepared from diagrams accurately constructed in accordance with the definitions of the curves discussed. For a number of these diagrams we are indebted to Cadet Midshipman J. H. Fillmore. They were first published in Part III. of the preliminary edition of this work, printed, for the use of the cadets at the U. S. Naval Academy, at the Government Printing Office, Washington, D. C., 1876.

The curve passes through the origin and through the point (a, a), for all values of n. Since $n > 1$, the curve is tangent to the axis of x at the origin.

254. The following three diagrams represent forms which the curve takes for different values of n. When n denotes a fraction, it is supposed to be reduced to its lowest terms.

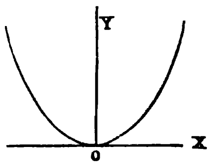


FIG. 31.

Fig. 31 represents the general shape of the curve when n is an even integer, or a fraction having an even numerator and an odd denominator.

Fig. 32 represents the form of the curve when n is an odd integer or a fraction with an odd numerator and an odd denominator, the origin being a point of inflexion.

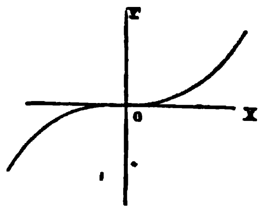


FIG. 32.

Fig. 33 represents the form of the curve when n is a fraction having an odd numerator and an even denominator. In this case y is regarded as a two-valued function, and is imaginary when x is negative.

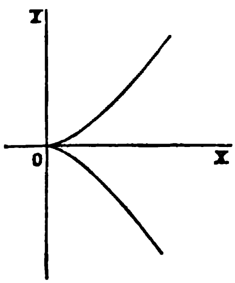


FIG. 33.

Fig. 31 is constructed for the parabola in which $n = 4$.

Fig. 32 is the *cubical parabola* in which $n = 3$.

Fig. 33 is the *semi-cubical parabola* in which $n = \frac{3}{2}$; the equation being

$$a^{\frac{1}{2}} y = \pm x^{\frac{3}{2}},$$

or

$$ay^2 = x^3.$$

The curves corresponding to the general equation

$$y = A + Bx + Cx^2 + Dx^3 + \ldots Lx^n$$

are sometimes called *parabolic curves* of the nth degree.

The Cissoid of Diocles.

255. Let A be a point on the circumference of a circle, and BC a tangent at the opposite extremity of the diameter AB; let AC be any straight line through A, and take $CP = AD$; then the locus of P is the *cissoid.*

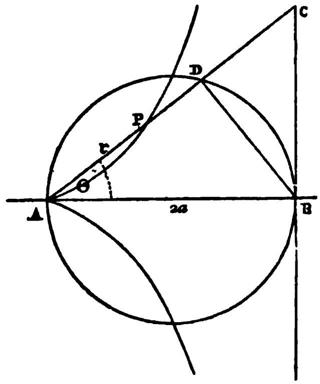


FIG. 34.

To find the polar equation, AB being the initial line, let DB be drawn, and denote the radius of the circle by a; then $AC = 2a \sec\theta$; and since ADB is a right angle, $AD = 2a\cos\theta$. The polar equation of the locus of P, A being the pole, is, therefore,

$$r = 2a(\sec\theta - \cos\theta) = 2a\frac{1-\cos^2\theta}{\cos\theta},$$

or
$$r = 2a\frac{\sin^2\theta}{\cos\theta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

256. To obtain the rectangular equation, we employ the equations of transformation

$$\sin\theta = \frac{y}{r}, \quad \cos\theta = \frac{x}{r}, \quad r^2 = x^2 + y^2;$$

whence, eliminating θ we obtain

$$r = 2a\frac{y^2}{rx},$$

and thence the rectangular equation of the curve

$$x(x^2 + y^2) = 2ay^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

or
$$y^2 = \frac{x^3}{2a - x}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The Cissoid Applied to the Duplication of the Cube.

257. The Greek mathematician Diocles, who is supposed to have lived in the sixth century A.D., was the inventor of this curve, which was employed by him in constructing the solution of the problem of *finding two mean proportionals*, of which the *duplication of the cube* is a particular case.

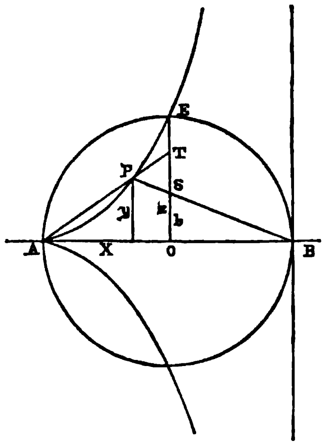


FIG. 35.

In Fig. 35, OE is a radius perpendicular to AB, and OS is a given length denoted by b; let BS intersect the curve in P, and let AP intersect OE in T; then, denoting OT by z, we have, by similar triangles,

$$\frac{2a - x}{a} = \frac{y}{b}, \text{ and } \frac{a}{x} = \frac{z}{y}.$$

By combining these equations with the rectangular equation of the cissoid, we deduce

$$z = \sqrt[3]{(a^2 b)}.$$

The above construction may be used to find the first of two mean proportionals between two given quantities a and b; for, if we insert two geometrical means between a and b, we have

$$a : \sqrt[3]{(a^2 b)} : \sqrt[3]{(ab^2)} : b.$$

In the duplication of the cube, we have $b = 2a$, therefore

$$z = a\sqrt[3]{2}, \text{ or } z^3 = 2a^3.$$

Example.

1. Show, by means of the rectangular equation of the cissoid, that it has a *cusp* at the origin, and an asymptote; also find the inclination at the point (a, a).

The Conchoid of Nicomedes.

258. Let A be a given point and BC a given straight line. On any line AC take CP, and also CP', equal to a given constant quantity denoted by b; the locus of P and P' is the *conchoid*.

A is the *pole*, BC the *directrix*, and b the *parameter*.

The locus of P is called the *inferior branch*, and that of P' the *superior branch*.

Denoting AB by a, we have, for the polar equation,

$$r = a \sec \theta \pm b. \quad . \quad . \quad (1)$$

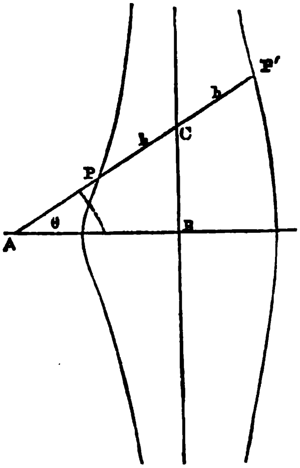


FIG. 36.

259. To obtain the rectangular equation, we put

$$\sec \theta = \frac{r}{x}, \text{ and } r = \sqrt{(x^2+y^2)};$$

whence

$$\sqrt{(x^2 + y^2)}(x - a) = \pm bx, \quad (2)$$

or $$(x^2 + y^2)(x-a)^2 = b^2x^2. \quad (3)$$

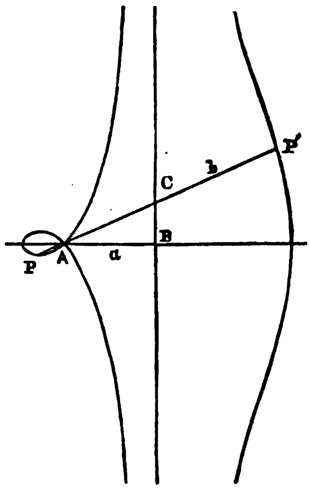


FIG. 37.

Equation (3), being equivalent to both equations (2), represents both branches of the curve.

When $b > a$, the inferior branch of this curve has a double point at A and a loop on the left, as shown in Fig. 37.

When $b = a$, the point A becomes a cusp.

Nicomedes, the inventor of this curve, was a Greek mathematician of the second century A.D.

The Trisection of an Angle by means of the Conchoid.

260. The conchoid, like the cissoid, was employed in constructing the solution of the problem of two mean proportionals; it was also applied to the trisection of an angle, another celebrated problem of the ancient geometers.

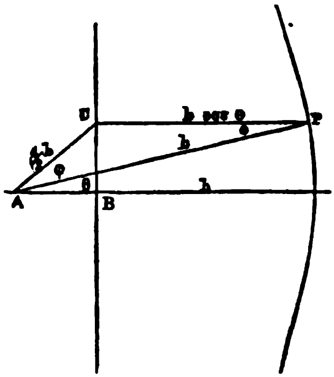


FIG. 38.

Let CAB be the given acute angle, and let CB be drawn perpendicular to AB. Construct the conchoid of which A is the pole, BC the directrix, and $2AC$ the parameter. Let CP parallel to AB intersect the curve in P; then PAB is $\frac{1}{3}$ of CAB; for, denoting CAP by φ, we have

$$\frac{\sin\varphi}{b\cos\theta} = \frac{\sin\theta}{\frac{1}{2}b};$$

whence $$\sin\varphi = 2\sin\theta\cos\theta = \sin 2\theta;$$

therefore $$\theta = \tfrac{1}{2}\varphi = \tfrac{1}{3}\,CAB.$$

Example.

1. Find, from the equation of the conchoid, the tangents at the origin, and thence determine when the curve has a crunode, when an acnode, and when a cusp.

The Quadratrix of Dinostratus.

261. Let the radius of the circle in Fig. 39 revolve uniformly, completing the semicircle AEB in the same time as

that required by the ordinate RP to move uniformly over the diameter; the intersection P of the ordinate and the radius will then describe the curve known as the *quadratrix*.

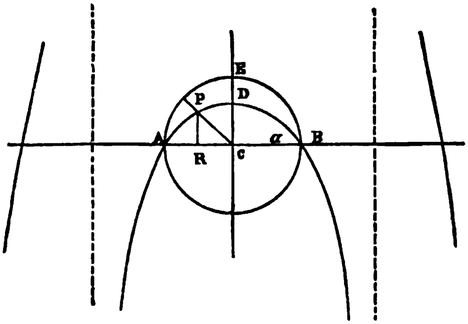


FIG. 39.

Denoting the radius by a, the angle at the centre by θ, and taking the origin at A, we have, by the mode of construction,

$$\frac{\theta}{\pi}=\frac{x}{2a}, \quad \text{and} \quad y=(a-x)\tan\theta.$$

Eliminating θ, we obtain

$$y=(a-x)\tan\frac{\pi x}{2a},$$

the equation of the quadratrix.

262. It is evident that, if AR be divided into any number of equal parts, and ordinates be erected at the points thus determined, the corresponding radii of the circle will divide the angle ACP into the same number of equal parts. Hence, by means of this curve, an angle may be divided into any number of equal parts. The curve was employed for this purpose by Dinostratus, a disciple of Plato; he also employed it in the quadrature of the circle. The latter application, from which

was derived the name of the curve, depends upon the result deduced below.

By evaluation, we obtain

$$CD = (a - x) \tan \frac{\pi x}{2a}\Big]_0 = \frac{2a}{\pi};$$

hence we have

$$\frac{AB}{CD} = \pi.$$

The Witch of Agnesi.

263. Given a point A on the circumference of a circle, and a tangent at the opposite extremity of the diameter AB; if the ordinate DR be produced, so that $PR = BC$, the locus of P will be the *witch*.

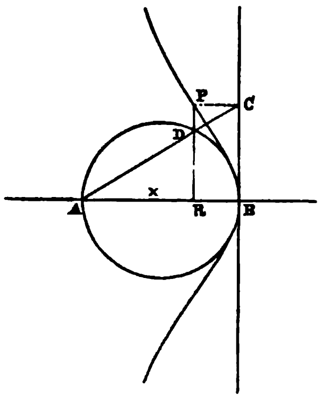


FIG. 40.

Taking the origin at A, and denoting the radius of the circle by a, we have

$$DR^2 = x(2a - x);$$

also, by the construction of the locus,

$$\frac{y}{2a} = \frac{DR}{x};$$

therefore

$$xy^2 = 4a^2(2a - x)$$

is the rectangular equation of the curve.

This curve was given under the name of *versiera* in a treatise on Analytical Geometry by Donna Maria Agnesi, an Italian mathematician of the eighteenth century.

Examples.

1. Show, by means of the equation of the witch, that the axis of y is an asymptote, and that the curve has two imaginary asymptotes parallel to the axis of x.

2. Find the points of inflexion in the witch, and the inclination of the curve at these points.

$$\left(\tfrac{3}{2}a,\ \pm\ \tfrac{2}{3}\sqrt{3}a\right) : \left.\frac{dy}{dx}\right]_{\frac{3}{2}a} = \mp\frac{8}{9}\sqrt{3}.$$

The Folium of Descartes.

264. This name has been given to a cubic defined by the equation

$$x^3 + y^3 - 3axy = 0.$$

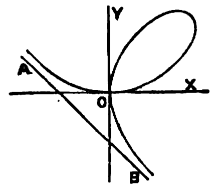


FIG. 41.

The form of this curve is represented in Fig. 41; the axes are tangent to the curve at the origin. By the method employed in Art. 228, the equation of the asymptote is found to be

$$x + y + a = 0.$$

Examples.

1. Show that the asymptote does not meet the curve, and find the point at which the ordinate is a maximum. $(a\sqrt[3]{2},\ a\sqrt[3]{4})$.

2. Show that, when the axes are turned through an angle of 45°, the equation of the folium is

$$x^3 + 3xy^2 - \tfrac{3}{2}\sqrt{2}\,.\,a\,(x^2 - y^2) = 0.$$

By means of this equation, find the asymptote and the point at which the ordinate is a maximum.

$$\left[\tfrac{1}{2}\sqrt{6}a,\ \tfrac{1}{2}\sqrt[4]{3}(\sqrt{3} - 1)\,a\right].$$

The Strophoid or Logocyclic Curve.

265. Let AB be perpendicular to the line BC, and denote its length by a. Let AC be any straight line through A, and take CP and CP' each equal to CB; then the locus of P and P' will be a continuous curve called the *strophoid.*

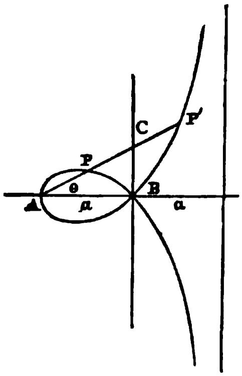


FIG. 42.

Since $AC = a \sec\theta$, and $CB = a\tan\theta$, the polar equation of the curve is

$$r = a(\sec\theta \pm \tan\theta). \quad . \quad . \quad (1)$$

As θ approaches $90°$, P approaches indefinitely near to A, while P' describes the infinite branch BP'. Since the perpendicular distance of P' from the line BC is the same as that of P, it is evident that the curve will have an asymptote parallel to BC at the distance a from it on the right.

This curve was styled the logocyclic curve by Dr. Booth, who discussed its properties elaborately in a paper read before the Royal Society, June 10, 1858.

266. To derive the rectangular equation of the curve, we substitute in equation (1)

$$\sec\theta = \frac{r}{x}, \quad \text{and} \quad \tan\theta = \frac{y}{x};$$

whence
$$r = a\left(\frac{r}{x} \pm \frac{y}{x}\right),$$

or
$$r\left(1 - \frac{a}{x}\right) = \pm\frac{ay}{x}.$$

Squaring, $$(x^2+y^2)\left(1-\frac{a}{x}\right)^2=\frac{a^2y^2}{x^2},$$

or $$x^2\left(1-\frac{a}{x}\right)^2+y^2\left(1-\frac{a}{x}\right)^2=\frac{a^2y^2}{x^2};$$

reducing $$(x-a)^2+y^2-\frac{2ay^2}{x}=0;$$

$$\therefore \qquad x(x-a)^2+y^2(x-2a)=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This is the rectangular equation of the strophoid referred to A, Fig. 42, as the origin.

267. Transferring the origin to the point B $(a, 0)$, we have

$$x^2(x+a)+y^2(x-a)=0,$$

or $$x(x^2+y^2)+a(x^2-y^2)=0; \quad . \quad . \quad . \quad . \quad . \quad (3)$$

whence it is evident that the tangents to the curve at the node are the lines

$$y=x \text{ and } y=-x.$$

From (3), by passing to polar coordinates, we derive the equation

$$r^3\cos\theta+ar^2(\cos^2\theta-\sin^2\theta)=0,$$

the factor r^2 indicating the node at the pole B. Rejecting this factor, we have

$$r\cos\theta+a\cos 2\theta=0.$$

Reversing the direction of the initial line by putting $\theta+180°$ in place of θ, we have

$$r = a\frac{\cos 2\theta}{\cos\theta}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

the polar equation of the strophoid, B being the pole and BA the initial line.

Examples.

1. Find the position of the maximum ordinate of the strophoid.

The origin being A, $x = \frac{1}{2}(3 - \sqrt{5})a$.

2. Let A be a fixed point in the circumference of a circle, and AB any chord passing through it; draw a diameter parallel to AB, and through B a line parallel to the tangent at A; prove that the locus of the intersection of this line with the diameter is the strophoid.

The Limaçon of Pascal.

268. If through a given point A on the circumference of a circle a line be drawn cutting the circumference again at C, and from the latter point a given distance be laid off in each direction on this line, the locus of the points thus determined is called the *limaçon.*

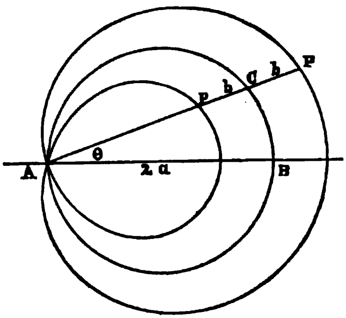


FIG. 43.

Let the diameter of the circle ACB, Fig. 43, be denoted by $2a$, and the given constant by b; then the polar equation of the locus of P and P' will be

$$r = 2a\cos\theta \pm b. \quad . \quad . \quad (1)$$

It is to be observed that each of these equations gives the entire curve; for, if we put $\alpha + 180°$ for θ, and use the *lower* sign, we obtain the point defined by

$$\theta = \alpha + 180°, \quad \text{and} \quad r = -2a\cos\alpha - b;$$

but this is the same as the point determined by

$$\theta = \alpha, \quad \text{and} \quad r = 2a\cos\alpha + b,$$

the latter being obtained by using the *upper* sign in equation (1).

Reversing the direction of the initial line, we have

$$r = b - 2a\cos\theta, \quad . \quad (2)$$

another form of the polar equation.

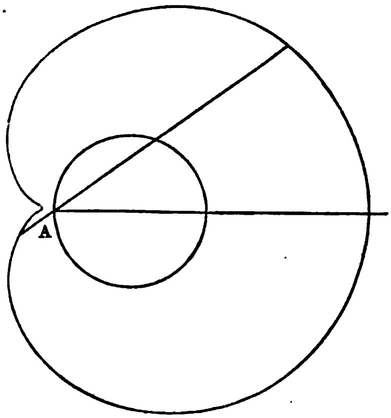


FIG. 44.

269. Transforming (1) to rectangular coordinates, we have

$$r = 2a\frac{x}{r} \pm b;$$

whence $$x^2 + y^2 - 2ax = \pm b\sqrt{(x^2 + y^2)},$$

or $$(x^2 + y^2)^2 - (4ax + b^2)(x^2 + y^2) + 4a^2x^2 = 0. \quad . \quad . \quad . \quad (3)$$

When $b > 2a$, the curve takes the form indicated in Fig. 44.

The limaçon occurs as a particular case of the Cartesian ovals (see Art. 275, Ex. 3); of the epitrochoid (see Art. 296); and of the hypotrochoid (see Art. 299).

270. The limaçon in which $b = a$ has been used in the trisection of an angle, and is hence called *the trisectrix*. Its polar equation is

$$r = a(2\cos\theta \pm 1),$$

and putting $b = a$ in (3), we have for its rectangular equation

$$(x^2 + y^2)^2 - 4ax(x^2 + y^2) + 3a^2x^2 - a^2y^2 = 0.$$

Examples.

1. In the case of the limaçon, express x in terms of θ, and thence determine the minimum abscissas of the curve. Derive the condition which makes the double tangent impossible.

$$\cos\theta = \mp\frac{b}{4a}; \text{ impossible when } b > 4a.$$

2. Determine, in a similar manner, the value of θ corresponding to the maximum ordinates.

$$\cos\theta = \frac{\mp b + \sqrt{(32a^2 + b^2)}}{8a}.$$

The Cardioid.

271. This curve is a particular case of the limaçon, in which $b = 2a$, the node at A becoming a cusp.

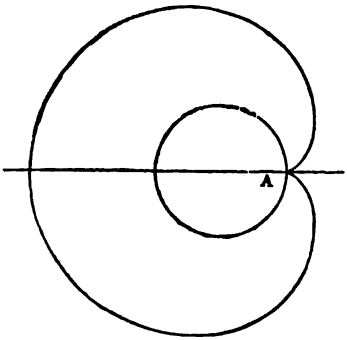


Fig. 45.

Putting $b = 2a$ in equation (1), Art. 268, we obtain, for the polar equation of the cardioid

$$r = 2a(\cos\theta \pm 1),$$

and, from equation (2),

$$r = 2a(1 - \cos\theta), \quad . \quad (1)$$

the position of the curve being that indicated in Fig. 45. This equation may also be written in the form

$$r = 4a \sin^2 \tfrac{1}{2}\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Transforming (1) to rectangular coordinates, we obtain the rectangular equation of the cardioid;

$$(x^2+y^2)^2 + 4ax(x^2+y^2) - 4a^2y^2 = 0. \quad . \quad . \quad . \quad (3)$$

This curve occurs as a particular case of the epicycloid, see Art. 296; and also of the hypocycloid, see Art. 299.

Example.

1. Determine the minimum abscissa, and the maximum ordinate of the cardioid.

Min. abscissa when $\theta = 60°$ and $r = a$:
Max. ordinate when $\theta = 120°$ and $r = 3a$.

The Cartesian Ovals.

272. If a point move in such a way that fixed multiples of its distances from two fixed points have a constant sum or difference, the path described will be a *Cartesian oval.* In other words, if we denote the distances of a moving point P from two fixed points A and B by r and r', and the distance AB by c, the locus of P will be a Cartesian oval when r and r' are connected by the linear relation

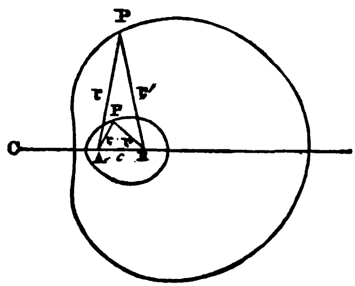


FIG. 46.

$$lr \pm mr' = \pm nc, \quad . \quad (1)$$

in which l, m, and n denote numerical constants.

The line AB is an axis of symmetry, since points symmetrically situated with reference to this line have the same values of r and r'.

Inasmuch as any two of the four equations included in (1) may be regarded as differing in the sign of only one term, it is evident that the resulting curves cannot intersect, except in the particular case when r or r' admits of the value zero; that is, when A or B is a point of the locus.

Moreover, an infinitely distant point cannot satisfy equation (1) if l and m have different values; hence in general the entire locus of this equation consists of closed branches or ovals which do not intersect.

273. To derive the polar equation, A being the pole and PAB the vectorial angle θ, we have the relation

$$r'^2 = r^2 + c^2 - 2cr\cos\theta,$$

and eliminating r' between this equation and (1), we obtain

$$(l^2 - m^2)r^2 + 2c(m^2\cos\theta \pm ln)\,r + (n^2 - m^2)c^2 = 0. \quad . \quad . \quad . \quad (2)$$

The two equations included in (2) determine the same points; for, when we use the lower sign and put $180° + \theta$ in place of θ, the sign of the coefficient of r is changed, and consequently the signs of both roots of the equation are changed. Hence the double sign in equation (2) may be omitted, and assuming l and m to have different values it may be written in the form

$$r^2 - (2a\cos\theta + b)r + k = 0, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in which

$$a = \frac{m^2c}{m^2 - l^2}, \qquad b = \frac{2lnc}{m^2 - l^2}, \qquad k = c^2\frac{m^2 - n^2}{m^2 - l^2}. \quad . \quad . \quad (4)$$

274. By transformation of coordinates we obtain

$$(x^2 + y^2 - 2ax + k)^2 = b^2(x^2 + y^2), \quad . \quad . \quad . \quad . \quad (5)$$

the rectangular equation of the Cartesian referred to A as an origin.

Putting $x = 0$ in equation (5) we have

$$y^2 + k = \pm by;$$

whence $$y = \pm \tfrac{1}{2}b \pm \tfrac{1}{2}\sqrt{(b^2 - 4k)}.$$

Hence a perpendicular to AB through A [the axis of y in equation (5)] cuts the curve in four real points when $b^2 - 4k$ is positive; substituting the values of b and k from equations (4) we find that $b^2 - 4k$ is positive when

$$l^2 + n^2 - m^2 \quad . \; . \; . \; . \; . \; . \; . \quad (6)$$

is positive, therefore *A is within both ovals when this expression is positive.* Since interchanging r and r' in equation (1) is equivalent to interchanging l and m, we infer that a perpendicular through B cuts the curve when

$$m^2 + n^2 - l^2 \quad . \; . \; . \; . \; . \; . \; . \quad (7)$$

is positive, and hence that B is within both ovals when this expression has a positive value. Now, since the sum of the expressions (6) and (7) is $2n^2$, which is always positive, it is impossible that both these expressions should be negative; it follows that at least one of the points A and B is within both ovals, and, consequently, (since they cannot intersect) that one oval is entirely within the other as represented in Fig. 46.

Equations (3) and (5) do not always represent Cartesians as defined above, since the values of the ratios of l, m, and n, determined by substituting in equations (4) given values of a, b, and k, may be imaginary.

275. These curves were first investigated by Descartes. Among the properties of the Cartesian ovals, one of the most noteworthy is the discovery of M. Chasles; namely, the exist-

ence of a third point C on the axis AB, such that a linear relation exists between the distances of a point of the curve from any two of the three points A, B, and C, which are called the foci. Thus, if we denote the distance CP by r'', a relation of the form (1) exists between r and r'', and a similar relation between r' and r''. Two of the foci are always within both ovals, like A and B in Fig. 46, while the third focus is exterior to both ovals.

Examples.

1. Show that when $l = m$ the Cartesian becomes an ellipse or an hyperbola according as n is greater or less than m, A and B being the foci.

2. Putting $l = m$ and $\frac{m}{n} = e$, and $c = 2ae$, derive from equation (2) (Art. 273) the polar equation of the ellipse or hyperbola.

$$r = \frac{a(1-e^2)}{1 - e\cos\theta}.$$

3. Show that if $n = m$ the Cartesian becomes the limaçon

$$r = 2a\cos\theta + b,$$

and find, in terms of a and b, the value of the ratio $\frac{l}{m}$ when the limaçon is regarded as a Cartesian, and also the distance c between the node at A and the second focus B. (The third focus in this case coincides with A.)

$$\frac{l}{m} = \frac{b}{2a}; \quad c = a - \frac{b^2}{4a}.$$

4. Determine from equation (3) the values of θ and r when the radius vector becomes a tangent to either oval.

$$\cos\theta = \frac{-\frac{1}{2}b \pm \sqrt{k}}{a}, \text{ and } r = \sqrt{k}.$$

When these values are real; that is, when k *[see equation* (4)] *is positive, the pole is the exterior focus.*

5. Show that if equation (3) represents a limaçon referred to the focus which does not coincide with the node, we must have

$$k = (a + \tfrac{1}{3}b)^2.$$

Show also that the curve is the crunodal limaçon if, a and b having opposite signs, b numerically exceeds $2a$; the acnodal limaçon if b is between $-2a$ and zero; and that the curve is imaginary when b and a have the same sign.

The Cassinian Ovals.

276. The locus of a point the product of whose distances from two fixed points is constant is called *the Cassinian Oval*, from Cassini, the name of the first investigator of this curve.

Let A and B be the two fixed points, or *foci*; let the middle point of the line AB be taken as the pole; denote the distance AB by $2a$, and the distances of the moving point P from A and B, by ρ and ρ'; then

$$\rho^2 = a^2 + r^2 + 2ar\cos\theta,$$

and $$\rho'^2 = a^2 + r^2 - 2ar\cos\theta.$$

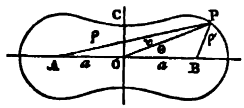


FIG. 47.

Whence, denoting the constant value of the product $\rho\rho'$ by c^2,

$$c^4 = (a^2 + r^2)^2 - 4a^2r^2\cos^2\theta,$$

or $$r^4 + 2a^2(1 - 2\cos^2\theta)\,r^2 + a^4 - c^4 = 0, \quad . \quad . \quad . \quad (1)$$

the polar equation of the *Cassinian*.

Transforming to rectangular coordinates, we obtain

$$(x^2 + y^2)^2 + 2a^2(y^2 - x^2) + a^4 - c^4 = 0. \quad . \quad . \quad . \quad (2)$$

The distance of the point C, at which the curve cuts the axis of y, from A or B, is by the definition of the curve equal

to c. In Fig. 47, c is taken greater than a; in this case, the curve consists of one continuous oval.

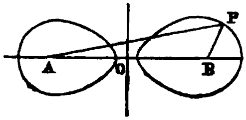


FIG. 48.

When $c < a$, the curve does not cut the axis of y, but consists of two distinct ovals, as in Fig. 48.

Examples.

1. Show that all the real points of the Cassinian are given by the equation

$$y^2 = -(x^2 + a^2) + \sqrt{(4a^2x^2 + c^4)},$$

and thence show that y is imaginary except when the numerical value of x falls between the limiting values $\sqrt{(a^2 + c^2)}$ and $\sqrt{(a^2 - c^2)}$. By means of this result, determine when the curve consists of two ovals. Show that when $c = 0$ the curve reduces to the two points A and B.

2. From the expression for $\frac{dy}{dx}$ derived from the equation of the Cassinian, show that the tangent is parallel to the axis of x where the curve cuts the axis of y, and also where it intersects the circle

$$x^2 + y^2 = a^2.$$

Derive from this result the equations of the double tangents to this curve.

$$y = \pm \frac{c^2}{2a}.$$

3. By means of the polar equation of the Cassinian, determine the values of θ which make r a tangent to the curve.

$$\sin 2\theta = \pm \frac{c^2}{a^2}.$$

The Lemniscata of Bernoulli.

277. This curve is a particular case of the Cassinian, in which $c = a$. The point C in this case falls at the origin, and becomes a crunode, as shown in Fig. 49.

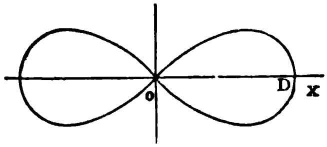


FIG. 49.

Making $c = a$ in equation (1) Art. 276, we have

$$r^4 + 2a^2(1 - 2\cos^2\theta)\,r^2 = 0;$$

or

$$r^2 = 2a^2 \cos 2\theta,$$

in which a denotes the distance from the centre to either focus.

The equation is usually written in the form

$$r^2 = a^2 \cos 2\theta, \quad \ldots\ldots \quad (1)$$

a here denoting the semi-axis of the curve; that is, OD the value of r when $\theta = 0$.

278. From (1), we have

$$r^2 = a^2(\cos^2\theta - \sin^2\theta),$$

or

$$r^2 = a^2\frac{x^2 - y^2}{r^2};$$

whence we have

$$(x^2 + y^2)^2 + a^2(y^2 - x^2) = 0, \quad \ldots\ldots \quad (2)$$

the rectangular equation of the lemniscata, referred to its centre and axes of symmetry.

If we turn the initial line back through 45°, (1) becomes

$$r^2 = a^2 \sin 2\theta, \quad \ldots\ldots \quad (3)$$

and the corresponding rectangular equation is

$$(x^2 + y^2)^2 = 2a^2xy. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

When the equation has this form, the coordinate axes are the tangents at the node.

Examples.

1. In the case of the lemniscata referred to its axes, determine by means of equation (1) the point at which the ordinate is a maximum.

$$\theta = 30^\circ\,; \quad r = \tfrac{1}{2}\sqrt{2}\,.\,a.$$

2. When the lemniscata is referred to the tangents at the node, determine by means of equation (3) the point at which y is a maximum.

$$\theta = 60^\circ\,; \quad r = \tfrac{1}{4}\sqrt{12}\,.\,a.$$

The Spiral of Archimedes.

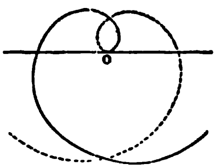


FIG. 50.

279. If the radius vector of a curve increase uniformly, while the vectorial angle also increases uniformly; that is, if

$$r = a\theta,$$

the curve generated will be the *spiral of Archimedes.*

The distance between two whorls, measured on a radius vector, is constant and equal to $2\pi a$.

The dotted portion of the curve in Fig. 50 is obtained by giving negative values to θ.

The Hyperbolic or Reciprocal Spiral.

280. This curve is defined by the polar equation

$$r = \frac{a}{\theta}.$$

When $\theta = 0$, r is infinite; hence there is a point at infinity in the direction of the initial line. To ascertain whether the curve has an asymptote, we have

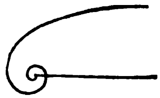


FIG. 51.

$$y = r \sin\theta = \frac{a \sin\theta}{\theta};$$

evaluating this fraction for $\theta = 0$, we have $y = a$; hence the line

$$y = a$$

is an asymptote. As θ increases, r decreases, the curve continually approaching but never reaching the pole. When θ is negative, r is negative and determines a similar curve, the complete curve being symmetrically situated with respect to a perpendicular to the initial line.

The Lituus.

281. This curve is defined by the polar equation

$$r^2\theta = a^2.$$

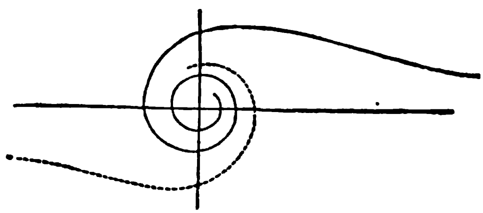


FIG. 52.

When θ is zero, r is infinite. To find the asymptote, we have

$$y = r \sin\theta = \frac{a \sin\theta}{\sqrt{\theta}}.$$

Evaluating, we find that $y = 0$ when $\theta = 0$; the initial line is, therefore, itself the asymptote.

The Logarithmic or Equiangular Spiral.

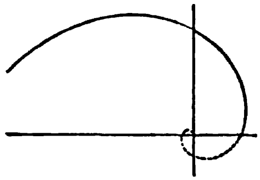


FIG. 53.

282. This spiral is defined by the polar equation

$$r = a\varepsilon^{n\theta}, \quad . \quad . \quad . \quad . \quad (1)$$

or

$$\log r = \log a + n\theta,$$

the logarithm of the radius vector being a linear function of the vectorial angle. The shape of the curve is indicated in Fig. 53.

It is proved in Art. 318 that this curve cuts its radius vector at a constant angle, hence it is sometimes called the *Equiangular Spiral.* A curve that cuts a system of lines or curves at a constant angle is called a *trajectory* of the system; hence this spiral is the trajectory of a system of straight lines passing through a common point. In (1) n is the cotangent of the constant angle, which is reckoned from the positive direction of the radius vector to the direction of the motion of the generating point when θ is increasing. In Fig. 53 this angle is acute, n being positive.

Example.

Prove that $r = a\varepsilon^{n\theta}$ and $r = b\varepsilon^{n\theta}$ represent the same spiral, the vectorial angles corresponding to equal values of r having a constant difference equal to $\frac{1}{n}\log\frac{a}{b}$.

The Loxodromic Curve and its Projections.

283. A trajectory of the meridians of any surface of revolution is called a *Loxodromic curve.*

The track of a ship when the course is uniform is a loxodromic curve traced upon a sphere, and is often called a *Rhumb line.*

If we project this curve stereographically upon the plane of the equator the meridians will project into straight lines, and, since in this projection angles are unchanged in magnitude, the projection of the curve will make a constant angle with the projections of the meridians and will therefore be an equiangular spiral.

Let θ denote the longitude of the generating point measured from the point at which the curve cuts the equator, and C the course; that is, the constant acute angle at which the curve cuts the meridians, the generating point being supposed to approach the pole as θ increases. Taking as the pole the projection of the pole of the sphere, the polar equation of the projected curve will be of the form

$$r = a\varepsilon^{n\theta}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which a is the radius of the sphere, since $\theta = 0$ gives $r = a$; we also have

$$n = -\cot C, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

since the angle whose cotangent is n is the supplement of C (see the preceding article).

Denoting by ϕ the co-latitude of the projected point we have, by the mode of projection,

$$\frac{r}{a} = \tan \tfrac{1}{2}\phi; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and, denoting the corresponding latitude by l,

$$\tfrac{1}{2}\phi = \tfrac{1}{4}\pi - \tfrac{1}{2}l.$$

Equation (1) is therefore equivalent to

$$\tan(\tfrac{1}{4}\pi - \tfrac{1}{2}l) = \varepsilon^{-\theta \cot C};$$

whence, solving for θ, we have

$$\theta = -\tan C \log_e \tan\left(\tfrac{1}{4}\pi - \tfrac{1}{2}l\right) = \tan C \log_e \tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}l\right),$$

or, employing common logarithms and expressing θ in degrees,

$$\theta^\circ = 131.9284 \tan C \cdot \log_{10} \tan\left(45^\circ + \tfrac{1}{2}l\right). \quad . \quad . \quad (4)$$

284. *The equation of the orthographic projection of the loxodromic curve on the plane of the equator* may be obtained in the following manner.

Denoting the radius vector of this projection of the curve by ρ, we have

$$\rho = a \sin \phi = a \cos l. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

To obtain a relation between ρ and θ we first derive a relation between r and ρ; thus, from equation (3) we have

$$\frac{r}{a} = \tan \tfrac{1}{2}\phi, \quad \text{and} \quad \frac{a}{r} = \cot \tfrac{1}{2}\phi;$$

hence
$$\frac{r}{a} + \frac{a}{r} = \frac{1}{\sin \frac{1}{2}\phi \cos \frac{1}{2}\phi} = \frac{2}{\sin \phi};$$

therefore, by equation (5),

$$\frac{r}{a} + \frac{a}{r} = \frac{2a}{\rho};$$

and, eliminating r by means of equation (1), we have

$$2a = \rho\left(\varepsilon^{n\theta} + \varepsilon^{-n\theta}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Passing to rectangular coordinates, we obtain

$$2a = \sqrt{(x^2 + y^2)}\left(\varepsilon^{n \tan^{-1}\frac{y}{x}} + \varepsilon^{-n \tan^{-1}\frac{y}{x}}\right).^{*} \quad . \quad . \quad . \quad (7)$$

* This curve is one of *Cotes' Spirals*. For a discussion of these spirals, see *Dynamics of a Particle*, by Tait and Steele, pp. 147–150, Fourth Edition, London, 1878.

The Parabolic Spiral.

285. If the axis of the parabola

$$y^2 = 4cx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

be conceived to be wrapped round a circle whose radius is a, and the ordinates corresponding to each point be laid off in the direction of the radius of the circle, the curve thus determined will be the *parabolic spiral.*

Taking the pole at the centre of the circle, and the radius passing through the vertex of the parabola as the initial line, we have

$$x = a\theta \quad \text{and} \quad y = r - a.$$

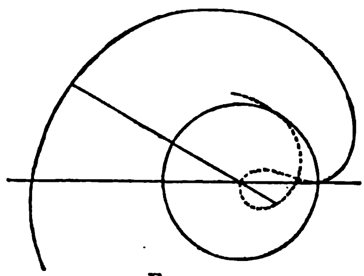


FIG. 54.

Substituting these values of x and y in equation (1), we obtain the polar equation

$$(r - a)^2 = 4ca\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The curve consists of two branches; the one determined by the positive values of $r - a$ is an infinite spiral without the circle; the other branch passes through the centre when $\theta = \dfrac{a}{4c}$, and emerges from the circle at the point at which

$$r = -a \quad \text{and} \quad \theta = \frac{a}{c}.$$

The Logarithmic or Exponential Curve.

286. The curve defined by the equation

$$y = \varepsilon^x$$

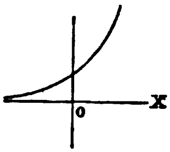


FIG. 55.

is called the *exponential curve;* and, since the equation can be written in the form

$$x = \log y,$$

it is also sometimes called the *logarithmic curve.* The axis of x is an asymptote since $\varepsilon^{-\infty} = 0$.

The curve defined by

$$y = a^x, \quad \text{or} \quad x = \log_a y,$$

is of the same general form. It passes through the point (0, 1) with an inclination whose tangent is $\log a$.

The Sinusoid.

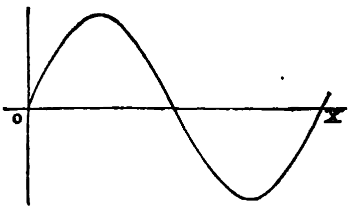


FIG. 56.

287. This curve is defined by the equation

$$y = b \sin \frac{x}{a}.$$

It consists of an infinite number of portions similar to the one shown in Fig. 56.

When $a = b = 1$, we have

$$y = \sin x.$$

The curve corresponding to this particular case may be distinguished as the *curve of sines.*

The Cycloid.

288. The path described by a point in the circumference of a circle which rolls upon a straight line is called a *cycloid*. The curve consists of an unlimited number of branches corresponding to successive revolutions of the generating circle; a single branch is, however, usually termed a cycloid.

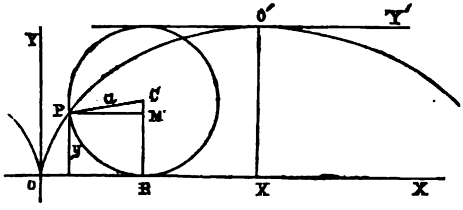


FIG. 57.

Let O, the point where the curve meets the straight line, be taken as the origin, let P be the generating point of the curve, and denote the angle PCR by ψ. Since PR is equal to the line OR over which it has rolled,

$$OR = PR = a\psi,$$

and, from Fig. 57, we readily derive

$$\left.\begin{aligned} x &= a(\psi - \sin\psi) \\ y &= a(1 - \cos\psi) \end{aligned}\right\} \quad \text{. . . . .} \quad (1)$$

289. These two equations express the values of x and y in terms of the auxiliary variable ψ, and constitute the equations of the cycloid. If desirable, ψ is easily eliminated from equations (1) and an equation between x and y obtained. Thus, from the second equation, we have

$$\cos\psi = \frac{a - y}{a}, \quad \text{whence} \quad \sin\psi = \frac{\sqrt{(2ay - y^2)}}{a};$$

and hence from the first of equations (1)

$$x = a\cos^{-1}\frac{a-y}{a} - \sqrt{(2ay - y^2)}, \quad . \quad . \quad . \quad . \quad (2)$$

or
$$x = a\,\text{vers}^{-1}\frac{y}{a} - \sqrt{(2ay - y^2)}.$$

Equations (1) will in general be found more convenient than equation (2). Thus we easily derive from (1)

$$\frac{dy}{dx} = \frac{\sin\psi\, d\psi}{(1-\cos\psi)\, d\psi} = \frac{\sin\psi}{1-\cos\psi};$$

whence

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\cos\psi - 1}{(1-\cos\psi)^2}\frac{d\psi}{dx} = -\frac{1}{a(1-\cos\psi)^2}.$$

290. The cycloid is frequently referred to the middle point O' or vertex of the curve as an origin, the directions of the axes being turned through 90°.

Denoting the coordinates referred to the axes $O'X'$ and $O'Y'$, in Fig. 57, by x' and y', we have

$$y' = x - a\pi = a(\psi - \pi - \sin\psi),$$
$$x' = 2a - y = a(1 + \cos\psi),$$

or, denoting $\psi - \pi$ by ψ',

$$\left.\begin{array}{l} y' = a(\psi' + \sin\psi') \\ x' = a(1 - \cos\psi') \end{array}\right\} . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In these equations $\psi' = 0$ gives the coordinates of the vertex and $\psi' = \pm\pi$ gives those of the cusps.

Example.

1. Prove that the chord RP joining the point of contact and the generating point is perpendicular to the tangent to the cycloid at P.

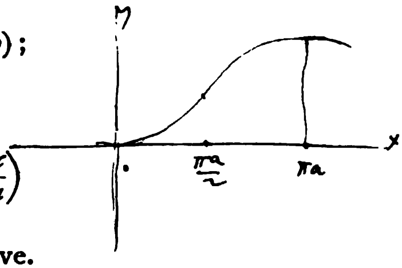
The Companion to the Cycloid.

291. This name was given by Roberval, one of the early investigators of the cycloid, to the curve described by the point M in Fig. 57.

The equations of the curve are, obviously,

$$x = a\psi,$$

and
$$y = a(1 - \cos\psi);$$



whence, eliminating ψ, we have

$$y = a\left(1 - \cos\frac{x}{a}\right)$$

for the rectangular equation of this curve.

Example.

1. Show that the companion to the cycloid is a sinusoid symmetrical to the line $y = a$, and that it bisects the area of the rectangle OO'. See Fig. 57.

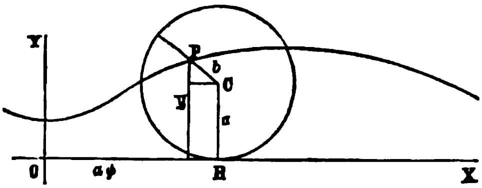
The area between the two curves regarded as generated by the variable line PM *parallel to* OX *is readily perceived to be equal to the area of the semicircle, which would be generated by this line were the point* M *to describe a line perpendicular to* OX. *In this way Roberval proved that the area of the cycloid is three times that of the generating circle.*

The Trochoid.

292. The *trochoid* is the general term applied to the curve described by *any* point in the radius of a circle rolling on a

straight line. If the generating point is taken within the rolling circle, the trochoid described is called the *prolate cycloid;* if without the circle, it is called the *curtate cycloid.*

FIG. 58.



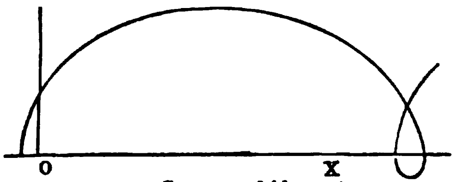
Prolate cycloid.

Denoting by b the distance of the generating point from the centre of the rolling circle, and employing the notation given in Fig. 58, the equations of the curve are

$$\left.\begin{array}{l} x = a\psi - b\sin\psi \\ y = a \;\; - b\cos\psi \end{array}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

When $b < a$, the curve is the *prolate cycloid;* when $b = a$, the *cycloid;* and, when $b > a$, the *curtate cycloid.*

FIG. 59.



Curtate cycloid.

Examples.

1. Show that the curtate cycloid cuts the axis of x at right angles, and, in general, that the line RP is perpendicular to the tangent to the trochoid at P.

2. Determine the points of inflexion in the prolate cycloid. Show that at the point of inflexion the radius of the generating circle is tangent to the curve. $\psi = \cos^{-1}\dfrac{b}{a}$.

To find $\dfrac{d^2y}{dx^2}$ *in terms of* ψ, *use the method exemplified in Art.* 289.

The Epicycloid.

293. When a circle, tangent to a fixed circle externally, rolls upon it, the path described by a point in the circumference of the rolling circle is called an *epicycloid*.

Taking the origin at the centre of the fixed circle, and the axis of x passing through A, (one of the positions of P when in contact with the fixed circle,) a, b, ψ, and χ, being defined by the diagram, we have, evidently,

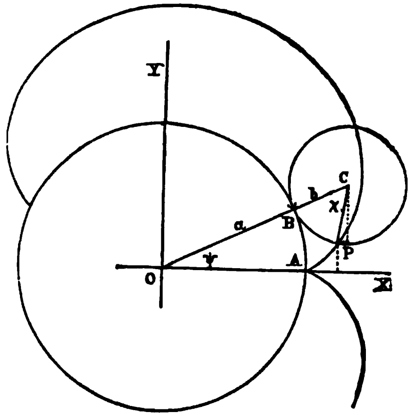


Fig. 60.

$$a\psi = b\chi \therefore \chi = \frac{a}{b}\psi.$$

The inclination of CP to the axis of x is equal to $\psi + \chi$, or to $\frac{a+b}{b}\psi$; the coordinates of P are found by subtracting the projections of CP on the axes from the corresponding projections of OC; hence

$$\left.\begin{aligned} x &= (a+b)\cos\psi - b\cos\frac{a+b}{b}\psi \\ y &= (a+b)\sin\psi - b\sin\frac{a+b}{b}\psi \end{aligned}\right\} \quad . \quad . \quad . \quad (1)$$

These are the equations of an epicycloid *referred to an axis passing through one of the cusps.*

Were the generating point taken at the opposite extremity of a diameter passing through P in the figure, the projection of CP would be *added* to that of OC; the axis of x would in this case pass through one of the *vertices* of the curve, and the second terms in the above values of x and y would have the *positive sign.*

Algebraic Forms of the Equations.

294. When a and b are incommensurable, the number of branches similar to that drawn in Fig. 60 is unlimited, and the curve is transcendental like the cycloid, since the number of points in which it may be cut by a straight line is unlimited. If, on the other hand, a and b are commensurable, another cusp will fall upon A after one or more circuits of the rolling circle, and the curve will begin to repeat itself. In this case, the curve will be algebraic, for the elimination of ψ will give an algebraic relation between x and y.

For example, when $b = \frac{1}{2}a$ the equations are

$$\left.\begin{aligned} x &= \tfrac{3}{2}a\cos\psi - \tfrac{1}{2}a\cos 3\psi \\ y &= \tfrac{3}{2}a\sin\psi - \tfrac{1}{2}a\sin 3\psi \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Employing the trigonometrical formulas

$$\cos 3\psi = 4\cos^3\psi - 3\cos\psi$$
$$\sin 3\psi = 3\sin\psi - 4\sin^3\psi,$$

we obtain

$$x = 3a\cos\psi - 2a\cos^3\psi = a\cos\psi\,[3 - 2\cos^2\psi]$$

$$y = 2a\sin^3\psi, \qquad \text{whence} \qquad \sin\psi = \left(\frac{y}{2a}\right)^{\frac{1}{3}};$$

and eliminating ψ

$$x^2 = a^2\left[1-\left(\frac{y}{2a}\right)^{\frac{2}{3}}\right]\left[1+2\left(\frac{y}{2a}\right)^{\frac{2}{3}}\right]^2 = a^2 + 3\frac{y^{\frac{2}{3}}a^{\frac{4}{3}}}{2^{\frac{2}{3}}} - y^2$$

or

$$4(x^2+y^2-a^2)^3 = 27y^2a^4. \quad . \quad . \quad . \quad . \quad (2)$$

Since this curve has two cusps, it is called the *two-cusped epicycloid.*

The remarks at the beginning of this article apply likewise to the curves defined in articles 295 and 297.

The Epitrochoid.

295. If a circle rolls upon a fixed circle, the curve described by any point in its plane is called an *epitrochoid.*

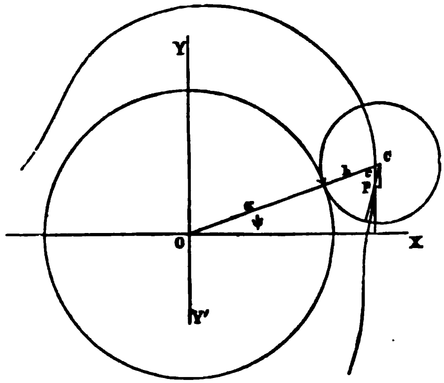


FIG. 61.

Let c denote the distance of the generating point from the centre of the rolling circle; and let ψ be defined as in Art. 293. To find the coordinates, we subtract the projections of c from the corresponding projections of OC. Whence

$$\left.\begin{aligned} x &= (a+b)\cos\psi - c\cos\frac{a+b}{b}\psi \\ y &= (a+b)\sin\psi - c\sin\frac{a+b}{b}\psi \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If c be greater than b the curve will contain loops, and will differ from that drawn in Fig. 61 as the curtate differs from the prolate cycloid.

296. When the fixed and the rolling circles are equal the epitrochoid becomes a limaçon, and the epicycloid becomes a cardioid. For, taking as the pole a point at the distance c on the right of the centre, it is obvious from Fig. 62 that $\theta = \psi$, and that

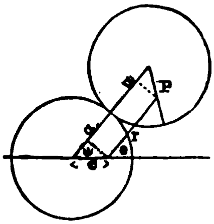


FIG. 62.

$$r = 2a - 2c \cos \theta,$$

which is the polar equation of the limaçon. See Eq. 2, Art. 268. When $c = a$ we have the equation of the cardioid. See Art. 271.

Examples.

1. Determine the points of inflexion in the epitrochoid.

$$\cos \chi = \frac{ac^2 + bc^2 + b^2}{bc(a + 2b)}.$$

2. The point of inflexion is impossible when the value of $\cos \chi$ determined in Ex. 1 exceeds unity; find the limiting values between which c must fall in order that the points of inflexion may be possible.

The limits are b and $\dfrac{b^2}{a + b}$.

3. Prove that, if $c = a + b$, the polar equation of the epitrochoid, OY' being the initial line, is

$$r = 2(a + b) \sin \frac{a}{a + 2b} \theta'.$$

The Hypocycloid and the Hypotrochoid.

297. When the rolling circle has internal contact with the fixed circle, the curve generated by a point on the circumference is called the *hypocycloid*, whether the radius of the rolling circle be greater or less than that of the fixed circle. Curves generated by points on the radius, either within or without the circumference of the rolling circle, are called *hypotrochoids*.

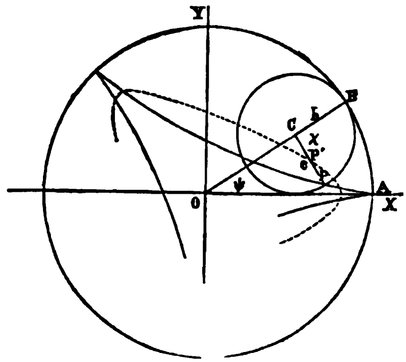


FIG. 63.

Adopting the notation used in deducing the equation of the epitrochoid, we have (see Fig. 63)

$$OC = a - b, \quad \text{and} \quad \chi = \frac{a}{b}\psi\,.$$

The inclination of CP to the negative direction of the axis of x is

$$\chi - \psi = \frac{a-b}{b}\psi;$$

hence the equations of the *hypotrochoid* are

$$\left.\begin{aligned} x &= (a-b)\cos\psi + c\cos\frac{a-b}{b}\psi \\ y &= (a-b)\sin\psi - c\sin\frac{a-b}{b}\psi \end{aligned}\right\}\quad . \; . \; . \; . \quad (1)$$

When $c = b$, we have the equations of the *hypocycloid*.

$$\left.\begin{aligned} x &= (a-b)\cos\psi + b\cos\frac{a-b}{b}\psi \\ y &= (a-b)\sin\psi - b\sin\frac{a-b}{b}\psi \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (2)$$

If the curve be referred to an axis passing through a *vertex*, the signs of the second terms in the values of x and y given above will be reversed as in the case of the epicycloid. See Art. 293.

298. When $b = \frac{1}{2}a$, equations (1) become

$$x = (b+c)\cos\psi \quad \text{and} \quad y = (b-c)\sin\psi;$$

whence, eliminating ψ, we have

$$\frac{x^2}{(b+c)^2} + \frac{y^2}{(b-c)^2} = 1,$$

the equation of an ellipse. Hence when the diameter of the rolling circle is half that of the fixed circle the hypotrochoid becomes an ellipse. If we put $c = b$, the above value of y becomes zero; hence in this case P (Fig. 63) describes the axis of x; that is, the diameter of the fixed circle.

It follows that all the points of the circumference of the rolling circle describe diameters of the fixed circle, we may therefore dispense with the circles and produce the same motion by constraining two points of a moving plane to describe intersecting straight lines on a fixed plane. The motion thus produced is known as *tram motion;* it follows that in tram motion all the points of the moving plane describe either straight lines or ellipses.

299. When the radius of the rolling circle is double that of the fixed circle as in Fig. 64 its centre C describes the circumference of the fixed circle. Let P be a point on the circumfer-

ence of the rolling circle, and A the point of the fixed circle with which it originally coincided; then, by the definition of the curve, the arc BP is equal in length to the arc BA, and, since its radius is double that of BA, the angle BCP is half the angle BOA. But BCA is half BOA; hence the angles BCP and BCA are equal, and consequently C, A, and P are on the same straight line as represented in the figure. Hence by the definitions given in Art. 268 and Art. 271 P' describes a limaçon and P describes a cardioid.

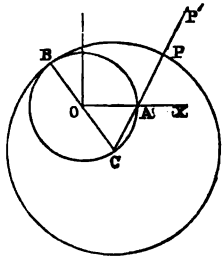


FIG. 64.

Examples.

1. Show that if $b > a$ in the case of the hypotrochoid, the curve may also be generated as an epitrochoid.

Put $\frac{a-b}{b}\psi = -\psi'$; *then* $\psi = \frac{b}{b-a}\psi'$. The constants for the curve as an epitrochoid are $a' = \frac{ac}{b}$, $b' = \frac{c}{b}(b-a)$, and $c' = b - a$.

2. Show that when $b < a$, in the case of the hypotrochoid, the curve may be generated as an hypotrochoid with other values of the constants.

Put $\frac{a-b}{b}\psi = \psi'$; *then* $\psi = \frac{b}{a-b}\psi'$. The new constants are $a' = \frac{ac}{b}$, $b' = \frac{c}{b}(a-b)$, and $c' = a - b$. Since $\frac{b'}{a'} = \frac{a-b}{a}$, we have

$$\frac{b'}{a'} + \frac{b}{a} = 1;$$

hence, if, in one of the two modes of generation, the ratio of the radius of the rolling circle to that of the fixed circle exceeds one-half, in the other it is less than one-half.

The Four-Cusped Hypocycloid.

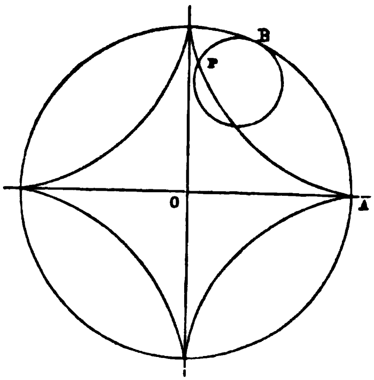


FIG. 65.

300. In the case of the hypocycloid when $b = \frac{1}{4}a$, the circumference of the rolling circle is one-fourth the circumference of the fixed circle, and the curve will have a cusp at each of the four points where the coordinate axes cut the fixed circle, as represented in Fig. 65.

On substituting $\frac{1}{4}a$ for b equations (2) Art. 297 become

$$\left.\begin{aligned} x &= \tfrac{3}{4}a\cos\psi + \tfrac{1}{4}a\cos 3\psi \\ y &= \tfrac{3}{4}a\sin\psi - \tfrac{1}{4}a\sin 3\psi \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (1)$$

Substituting the values of $\cos 3\psi$ and $\sin 3\psi$ from the formulas,

$$\cos 3\psi = 4\cos^3\psi - 3\cos\psi,$$

and
$$\sin 3\psi = 3\sin\psi - 4\sin^3\psi,$$

we have
$$\left.\begin{aligned} x &= a\cos^3\psi \\ y &= a\sin^3\psi \end{aligned}\right\}; \quad . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

whence
$$x^{\frac{2}{3}} = a^{\frac{2}{3}}\cos^2\psi,$$

and
$$y^{\frac{2}{3}} = a^{\frac{2}{3}}\sin^2\psi.$$

Adding, we have $$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}, \quad (3)$$

the rectangular equation of the curve. This equation, when freed from radicals, will be found to be of the sixth degree.

Example.

1. In the case of the four-cusped hypocycloid, express in terms of ψ the tangent of the inclination of the chord BP, Fig. 65, and prove that this chord is perpendicular to the tangent to the curve at the point P.

The Involute of the Circle.

301. The curve described by any point in a straight line which rolls upon a curve is called an *involute* of the given curve. The curves described by different points of the rolling tangent usually differ in shape, but when the given curve is a circle the involutes differ only in position.

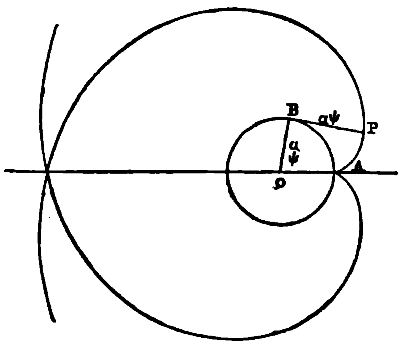


FIG. 66.

Let BP be one position of the rolling tangent to a circle, and let the axis of x pass through A, the position of the generating point when in contact with the circle, then

$$BP = a\psi.$$

By projecting the broken line OBP on the coordinate axes, we have, OBP being a right angle,

$$\left.\begin{aligned} x &= a\cos\psi + a\psi\sin\psi \\ y &= a\sin\psi - a\psi\cos\psi \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

If the tangent roll back beyond the initial position, a cusp will be formed at the point A, and the curve will consist of two symmetrically situated infinite branches, as in Fig. 66.

Example.

1. Show that the tangent to the involute of the circle is perpendicular to the rolling tangent, and find the maximum ordinate and abscissa in the first whorl.

Max. ordinate when $\psi = \pi$;
Max. abscissa when $\psi = \frac{1}{2}\pi$.

The Catenary.

302. The transcendental curve defined by the equation

$$y = \frac{c}{2}\left(\varepsilon^{\frac{x}{c}} + \varepsilon^{-\frac{x}{c}}\right)$$

is called the *catenary*, because it is the form assumed by a chain, or perfectly flexible cord of uniform weight per linear unit, when suspended from two fixed points.

The curve is evidently symmetrical with reference to the axis of y, and cuts it at the distance c above the origin.

This curve was first noticed by Galileo, but its true nature was first discovered by James Bernoulli.

The Tractrix.

303. If a heavy body situated at A on a horizontal plane be attached to a string of fixed length a, the other end being drawn along a straight line OX, it will describe a curve having the characteristic property that the intercept on the tangent between the point of tangency and the axis OX has a constant length.

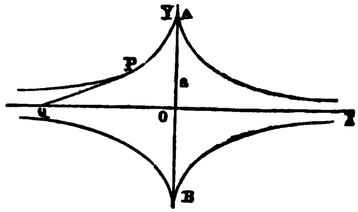


FIG. 67.

This curve is called the *tractrix* or *tractory*. The property mentioned above is expressed by the differential equation

$$\frac{dy}{dx} = \pm \frac{y}{\sqrt{(a^2 - y^2)}},$$

and the equation of the curve is

$$\pm x = a \log \frac{a + \sqrt{(a^2 - y^2)}}{y} - \sqrt{(a^2 - y^2)}.$$

The curve consists of four symmetrical infinite branches forming cusps at A and B, and asymptotic to the axis of x.

Curves of Pursuit.

304. If, while a point Q moves uniformly from O in the straight line OQ, P, starting from A, moves uniformly in the direction PQ, the path of P is called a *curve of pursuit*.

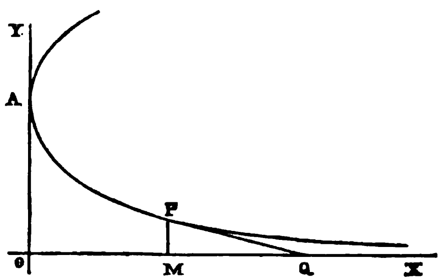


FIG. 68.

The characteristic property of these curves is that the length of the arc AP has a fixed ratio to OQ.

Denoting the constant ratio $\frac{OQ}{AP}$ by e, and supposing e not equal to unity, the equation is*

$$2\left(x - \frac{ae}{1 - e^2}\right) = \frac{y^{1+e}}{a^e(1 + e)} - \frac{a^e y^{1-e}}{1 - e}. \quad . \quad . \quad (1)$$

The axis of x is an asymptote or a tangent, according as e is greater or less than unity.

If $e = 1$, equation (1) takes an indeterminate form; in this case, the equation of the curve is

$$4ax + a^2 = y^2 - 2a^2 \log \frac{y}{a}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Roulettes.

305. When any curve rolls upon a fixed curve, every point in the plane of the rolling curve describes a curve in the fixed plane. Curves generated in this manner are termed *roulettes.*

The cycloid, the trochoids, and the epitrochoids are examples of roulettes in which the rolling curve is a circle. Involutes, as defined in Art. 301, are roulettes in which a straight line takes the place of the rolling curve.

Inverse Curves.

306. If a fixed point O in the plane of a given curve be taken as the pole, and on the radius vector OP a point P' be so taken that $OP \cdot OP'$ is constant, the curve described by P' is called the *inverse* of the given curve with reference to the point O.

* See *Dynamics of a Particle*, by Tait and Steele, Art. 32–36. Fourth edition, London, 1878.

The point O is called *the centre of inversion*, and the value of the constant product is called *the modulus*. It is evident that the curve described by P is also the inverse of that described by P'.

The equation of the inverse of any curve is readily derived from the polar equation of the curve referred to the centre of inversion. Thus, the polar equation of the cardioid referred to its cusp is (Art. 271)

$$r = 2a\,(1 - \cos\theta);$$

whence taking $4a^2$ as the modulus we have, for the polar equation of the inverse curve,

$$r = \frac{2a}{1 - \cos\theta},$$

the equation of a parabola referred to its focus.

Again the polar equation of the cissoid is (Art. 255)

$$r = 2a\,\frac{\sin^2\theta}{\cos\theta}.$$

Hence, taking the same modulus $4a^2$, we have

$$r = 2a\,\frac{\cos\theta}{\sin^2\theta};$$

or in rectangular coordinates

$$y^2 = 2ax,$$

the equation of a parabola referred to its vertex.

It follows therefore that the inverse of the parabola is a cardioid when the focus is the centre of inversion, and a cissoid when the vertex is the centre of inversion.

Examples.

1. Prove that, the modulus being a^2, the strophoid is its own inverse, when the vertex A, Fig. 42, is the centre of inversion; but that when its node is the centre of inversion its inverse is an equilateral hyperbola.

2. Prove that, when the node is taken as the centre of inversion, the inverse of the limaçon is a conic of which this point is a focus.

3. Prove that the Cartesian is its own inverse with respect to either focus.

4. Show that, when the Cassinian consists of two ovals as in Fig. 48, each oval may be regarded as its own inverse with reference to the centre O.

5. Show by means of equation (1) Art. 277, or by equation (3) Art. 278, that the inverse of the Lemniscata is the equilateral hyperbola having the same axis.

6. Prove that the inverse of the spiral of Archimedes is the hyperbolic spiral; and that that of the equiangular spiral is the same curve in a new position.

Pedals.

307. The locus of the foot of a perpendicular from a fixed point upon the tangent to a given curve is called *the pedal* of the given curve with reference to the fixed point. This point is called *the pedal origin.*

The pedals of a given curve are identical in form with the roulettes described when a curve equal to the given curve rolls upon it, the points of contact being corresponding points of the two curves. For the fixed and rolling curves are symmetrically situated with

reference to the common tangent; hence the straight line joining the point that describes the roulette with the corresponding point in the plane of the fixed curve is perpendicular to the common tangent, and double the length of the perpendicular whose extremity describes the pedal. The pedal is therefore similar to the roulette.

308. The pedal of a pedal with reference to the same origin is called the *second pedal* of the given curve; in like manner the third and higher pedals are formed. The curve of which a given curve is the pedal is called its *negative pedal.*

One of the methods of deriving the equation of a pedal will be found in Art. 327. See also Art. 350.

The method of deriving the equations of negative pedals will be found in Art. 381.

Reciprocal Curves.

309. If upon OR, the perpendicular from a fixed point O upon a straight line, a point P' be so taken that

$$OR \cdot OP' = k^2,$$

(k denoting a constant), the point P' is called the *pole* of the straight line with reference to the point O. The given line is called the *polar* of P'.

If now the straight line is the tangent at P to a given curve the locus of R is the pedal of this curve; that of P' is therefore *the inverse of the pedal.* It will be shown hereafter (Art. 384) that the locus of P bears the same relation to the locus of P'; that is to say, each of these curves is the locus of the pole of the tangent to the other curve; hence these curves are called *reciprocal polars*, or simply *reciprocals.**

*The pole of a straight line and the polar of a curve are frequently defined with reference to any conic. In the system above described, which is the one chiefly in use, the conic of reference becomes a circle whose centre is O and whose radius is k.

Methods of deriving the equation of the reciprocal will be found in Art. 385.

The Inverse and Pedal of the Conic.

310. It will be shown in Art. 386 that the reciprocal of a conic is another conic, hence it follows from the preceding article that the pedal of a conic is the inverse of another conic. The inverse and the pedal of the conic therefore constitute the same class of curves. These curves always have a node at the centre of inversion or pedal origin. They are generally of the fourth degree and constitute a subdivision of an important class of curves known as *Bicircular Quartics.* When the centre of inversion is on the curve, they are of the third degree and belong to the class called *Circular Cubics.*

The application of the Differential Calculus to Evolutes, Envelopes, and Caustics, will be found in Chapter X.

CHAPTER X.

APPLICATIONS OF THE DIFFERENTIAL CALCULUS TO PLANE CURVES.

XXXII.

The Equation of the Tangent.

311. THE equation of a curve being given in the form $y = f(x)$, the inclination of the tangent at any point is determined by the equation

$$\tan\phi = \frac{dy}{dx} = f'(x).$$

Hence, if (x_1, y_1) be a point of the curve, the equation of a tangent at (x_1, y_1) will be found by giving to the direction-ratio m, in the general equation

$$y - y_1 = m(x - x_1),$$

the value $\left.\frac{dy}{dx}\right]_{x_1}$; thus

$$y - y_1 = \left.\frac{dy}{dx}\right]_{x_1}(x - x_1), \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or

$$y - y_1 = f'(x_1)(x - x_1). \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For example, in the case of the semi-cubical parabola

$$y^3 = ax^2,$$

we have
$$\frac{dy}{dx} = \frac{2}{3}\sqrt[3]{\frac{a}{x}}.$$

The point (a, a) is a point of this curve; the equation of the tangent at this point is, therefore,

$$y - a = \tfrac{2}{3}(x - a),$$

or
$$3y - 2x = a.$$

If the given equation is not in the form $y = f(x)$, we can still use equation (1); but, since $\dfrac{dy}{dx}$ will then be expressed in terms of x and y, the value of the derivative must evidently be obtained by substituting in this expression the given simultaneous values of x_1 and y_1.

To illustrate, let the equation of the curve be

$$xy^3 - 3x^2y + 6y^2 + 2x = 0,$$

and let it be required to find the tangent at the point (2, 1), which is a point of the curve since these values of x and y satisfy the given equation.

Differentiating, we obtain

$$\frac{dy}{dx} = -\frac{y^3 - 6xy + 2}{3(xy^2 - x^2 + 4y)};$$

whence
$$\left.\frac{dy}{dx}\right]_{(2,\,1)} = \tfrac{3}{2}.$$

The equation of the tangent at this point is, therefore,

$$y - 1 = \tfrac{3}{2}(x - 2).$$

The Equation of the Normal.

312. A perpendicular to the tangent at its point of contact is called a *normal* to the curve.

The coordinate axes being rectangular, the direction-ratio of the normal is the negative reciprocal of that of the tangent; for the inclination of the normal is $\frac{1}{2}\pi + \phi$, and

$$\tan(\tfrac{1}{2}\pi + \phi) = -\cot\phi.$$

The equation of the normal may, therefore, be written thus—

$$y - y_1 = -\left.\frac{dx}{dy}\right]_{x_1}(x - x_1), \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or

$$y - y_1 = -\frac{1}{f'(x_1)}(x - x_1). \quad . \quad . \quad . \quad . \quad . \quad (2)$$

As an illustration, let us take the equation of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$$

whence

$$\frac{dy}{dx} = -\frac{b^2x}{a^2y}.$$

The equation of the normal at any point (x_1, y_1) of the ellipse is, therefore,

$$y - y_1 = \frac{a^2y_1}{b^2x_1}(x - x_1).$$

The Equation of the Tangent to a Conic.

313. The general equation of the second degree may be written in the form

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0;$$

from this we derive

$$\frac{dy}{dx} = -\frac{Ax + By + D}{Bx + Cy + E}.$$

The general equation of a line tangent to the curve at the point (x_1, y_1) is, therefore,

$$(y - y_1)(Bx_1 + Cy_1 + E) + (x - x_1)(Ax_1 + By_1 + D) = 0, \quad (1)$$

which reduces to

$$y(Bx_1 + Cy_1 + E) + x(Ax_1 + By_1 + D) \\ - (Ax_1^2 + 2Bx_1y_1 + Cy_1^2 + Dx_1 + Ey_1) = 0; \quad (2)$$

but, since (x_1, y_1) is a point of the curve, we have

$$Ax_1^2 + 2Bx_1y_1 + Cy_1^2 + 2Dx_1 + 2Ey_1 + F = 0, \quad . \quad . \quad . \quad (3)$$

and, by adding (2) and (3), we have

$$x(Ax_1 + By_1 + D) + y(Bx_1 + Cy_1 + E) + Dx_1 + Ey_1 + F = 0, \quad (4)$$

the equation of the tangent expressed in terms of the coordinates of the point of contact.

This equation may also be written in the form

$$Axx_1 + B(x_1y + xy_1) + Cyy_1 + D(x + x_1) + E(y + y_1) + F = 0.$$

Subtangents and Subnormals.

314. Denoting by s the length of the arc measured from some fixed point, $\frac{ds}{dt}$ denotes the velocity of P, the generating point

of the curve; let PT, equal to ds, be measured on the tangent at P, then PQ and QT will represent dx and dy, and the angle TPQ will be ϕ; hence

$$\cos\phi = \frac{dx}{ds}, \quad \sin\phi = \frac{dy}{ds}, \quad (1)$$

and $$ds = \sqrt{(dx^2 + dy^2)}. \quad . \quad . \quad (2)$$

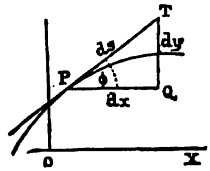


FIG. 69.

315. The distance PT (Fig. 70) on the tangent line intercepted between the point of contact and the axis of x is sometimes called the *tangent*, and in like manner the intercept PN is called the *normal*.

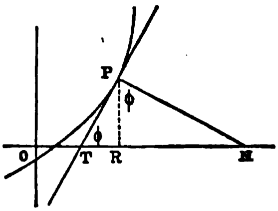


FIG. 70.

From the triangles PTR and NPR, we have

$$PT = y \operatorname{cosec}\phi = y\frac{ds}{dy} = y\sqrt{\left[1 + \left(\frac{dx}{dy}\right)^2\right]},$$

$$PN = y \sec\phi = y\frac{ds}{dx} = y\sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}.$$

The projections of these lines on the axis of x, that is TR and RN, are called the *subtangent* and the *subnormal*.

From the same triangles, we have

the subtangent, $\qquad TR = y\cot\phi = y\dfrac{dx}{dy},$

and the subnormal, $\quad RN = y\tan\phi = y\dfrac{dy}{dx}.$

The Perpendicular from the Origin upon the Tangent.

316. If a perpendicular p to the tangent PR be drawn from the origin, we have, from the triangles in Fig. 71,

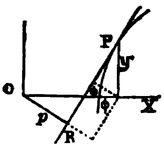


FIG. 71.

$$p = x \sin \phi - y \cos \phi, \quad . \quad . \quad . \quad (1)$$

$\phi - 90°$ *being taken as the positive direction of* p. Substituting the values of $\sin \phi$ and $\cos \phi$, equation (1) becomes

$$p = \frac{xdy - ydx}{ds} = \frac{xdy - ydx}{\sqrt{(dx^2 + dy^2)}}. \quad . \quad . \quad . \quad . \quad (2)$$

For example, let us determine p in the case of the four-cusped hypocycloid,

$$x = a \cos^3 \psi, \qquad y = a \sin^3 \psi.$$

Differentiating,

$$dx = -3a \cos^2 \psi \sin \psi \, d\psi, \quad \text{and} \quad dy = 3a \sin^2 \psi \cos \psi \, d\psi;$$

whence
$$ds = 3a \sin \psi \cos \psi \, d\psi.$$

Substituting in equation (2) we obtain

$$p = a \cos^3 \psi \sin \psi + a \sin^3 \psi \cos \psi = a \sin \psi \cos \psi = \sqrt[3]{(axy)}.$$

To ascertain the direction of p it is necessary to determine ϕ. The ambiguity in the value of ϕ as determined from the equation $\tan \phi = \dfrac{dy}{dx}$ may be removed by means of one of the formulas of Art. 314. Thus, in the present case, we have

$$\tan \phi = -\tan \psi, \quad \text{whence} \quad \phi = -\psi, \quad \text{or} \quad \phi = \pi - \psi;$$

but, since $$\cos\phi = \frac{dx}{ds} = -\cos\psi,$$

we must take $$\phi = \pi - \psi.$$

The direction of p when positive is therefore $\frac{1}{2}\pi - \psi$.

Examples XXXII.

1. In the case of the parabola of the nth degree

$$a^{n-1}y = x^n,$$

find the equations of the tangent and the normal at the point (a, a).

2. Find the subtangent and the subnormal of the parabola

$$y^2 = 4ax.$$

3. Prove that the normal to the catenary (Art. 302) is $\frac{y^2}{c}$.

4. Prove that the subtangent of the exponential curve

$$y = a^x$$

is constant, and find the ordinate of the point of contact when the tangent passes through the origin. ϵ.

5. Find the subnormal of the ellipse whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

6. Find the subtangent of the curve

$$a^{n-1}y = x^n;$$

also the subtangent of the curve

$$y^n = a^{n-1}x.$$

$\frac{x}{n}$, and nx.

7. In the case of the parabola

$$y^2 = 4ax,$$

find p in terms of x.

For the upper branch, $p = -\dfrac{x\sqrt{a}}{\sqrt{(a+x)}}$.

8. Find, in terms of ψ, the equation of the tangent to the four-cusped hypocycloid (Art. 300), and thence show that the part intercepted between the axes is of constant length.

9. Show that all the curves represented by the equation

$$\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2,$$

(different values being assigned to n), have a common tangent at the point (a, b); find the equation of this tangent.

10. Show that the equation of the tangent to the curve

$$\frac{x^{\frac{2}{3}}}{a^{\frac{2}{3}}} + \frac{y^{\frac{2}{3}}}{b^{\frac{2}{3}}} = 1,$$

at the point (x_1, y_1), is

$$a^{\frac{2}{3}}x_1^{\frac{1}{3}}y + b^{\frac{2}{3}}y_1^{\frac{1}{3}}x = a^{\frac{2}{3}}b^{\frac{2}{3}}x_1^{\frac{1}{3}}y_1^{\frac{1}{3}};$$

and, denoting the intercepts on the axes by x_0 and y_0, prove that

$$\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1.$$

11. Find the equation of the tangent at any point of the curve

$$x^n + y^n = a^n.$$

$$yy_1^{n-1} + xx_1^{n-1} = a^n.$$

12. Find the equation of the tangent at any point of the curve

$$x^m y^n = a^{m+n}.$$

$$nx_1 y + my_1 x = (m+n)x_1 y_1.$$

13. Find the general equation of a tangent to the conic

$$x^2 - 2y^2 - 4xy - x = 0.$$

Show that the point $(1, -2)$ is on the curve, and find the equation of the tangent line at this point.

14. Show that, when the conic passes through the origin, the equation of a tangent line at this point, as derived from the general equation, Art. 313, is identical with that obtained by the method of Art. 218.

15. In the case of the epicycloid, find the value of ds in terms of the auxiliary angle ψ. See Art. 293.

$$ds = 2(a + b)\sin\frac{a\psi}{2b}\,d\psi.$$

16. Determine the value of p in the case of the epicycloid employing the value of ds given in the preceding example.

$$p = (a + 2b)\sin\frac{a\psi}{2b}.$$

XXXIII.

Polar Coordinates.

317. When the equation of a curve is given in polar coordinates the vectorial angle θ is usually taken as the independent variable; hence, denoting by s an arc of the curve, it is usual to assume that ds and $d\theta$ have the same sign; that is, that $\dfrac{ds}{d\theta}$ is positive.

In Fig. 72 let PT, a portion of the tangent line, represent ds; then, producing r, let the rectangle PT be completed, and

let ψ denote the angle TPS; that is, the angle between the *positive directions* of r and s. The resolved velocities of P along and perpendicular to the radius vector are $\dfrac{dr}{dt}$ and $\dfrac{rd\theta}{dt}$, the latter being the velocity which P would have if r were constant; that is, if P moved in a circle described with r as a radius. Hence we have

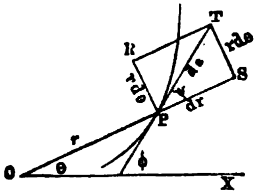


FIG. 72.

$$PS = dr \qquad \text{and} \qquad PR = rd\theta.$$

From the triangle PST, we derive

$$\tan\psi = \frac{rd\theta}{dr}, \qquad \sin\psi = \frac{rd\theta}{ds}, \qquad \cos\psi = \frac{dr}{ds}, \quad . \quad . \quad (1)$$

and

$$\frac{ds}{d\theta} = \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

318. The second of equations (1) shows that, in accordance with the assumption that ds has the sign of $d\theta$, the value of ψ will always be either in the first or in the second quadrant.

The first of equations (1) is equivalent to

$$\cot\psi = \frac{dr}{rd\theta}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which shows that $\cot\psi$ is *the logarithmic derivative of* r *regarded as a function of* θ. Thus in the case of the logarithmic spiral

$$r = a\epsilon^{n\theta}$$

we have $$\log r = \log a + n\theta,$$

hence $$\cot\psi = n$$

whence it follows that, in the case of this curve, ψ is constant. See Art. 282.

319. It is frequently convenient to employ in place of the radius vector its reciprocal, which is usually denoted by u; then

$$r = \frac{1}{u}, \quad \text{and} \quad dr = -\frac{du}{u^2}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Making these substitutions in equations (2) and (3) we have

$$\frac{ds}{d\theta} = \frac{1}{u^2}\sqrt{\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right]}. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and

$$\cot\psi = -\frac{du}{ud\theta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Polar Subtangents and Subnormals.

320. Let a straight line perpendicular to the radius vector be drawn through the pole, and let the tangent and the normal meet this line in T and N respectively; then the projections of PT and PN upon this line, that is OT and ON, are called respectively the ***polar subtangent*** and the ***polar subnormal.*** In Fig. 73, $OPT = \psi$; whence

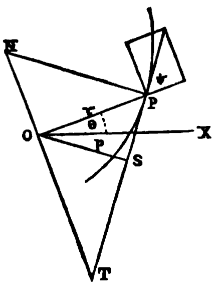


FIG. 73.

$$OT = r\tan\psi = r^2\frac{d\theta}{dr} = -\frac{d\theta}{du},$$

and

$$ON = r\cot\psi = \frac{dr}{d\theta} = -\frac{du}{u^2 d\theta}.$$

Fig. 73 shows that the value of OT is positive when its

direction is $\theta - 90°$; that of ON is, on the other hand, positive when its direction is $\theta + 90°$.

The Perpendicular from the Pole upon the Tangent.

321. Let p denote the perpendicular distance from the pole to the tangent; then, from Fig. 73 we obtain

$$p = r\sin\phi = r^2\frac{d\theta}{ds} = \frac{r^2}{\sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}}. \quad . \quad . \quad (1)$$

These expressions give positive values for p, because $\frac{ds}{d\theta}$ is assumed to be positive, and Fig. 73 shows that p has the direction $\phi - 90°$, ϕ being the angle which the positive direction of s makes with the initial line.

The relation between p and u is obtained thus:—from (1) we have

$$\frac{1}{p^2} = \frac{ds^2}{r^4 d\theta^2},$$

and, transforming by the formulas of Art. 319,

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

322. The expression deduced below for the function $u + \frac{d^2u}{d\theta^2}$ is frequently useful.

Differentiating (2), we have

$$2udu + 2\frac{du}{d\theta}\cdot\frac{d^2u}{d\theta} = -\frac{2dp}{p^3};$$

hence
$$\left(u + \frac{d^2u}{d\theta^2}\right) du = -\frac{dp}{p^2},$$

or since
$$du = -\frac{dr}{r^2},$$

$$u + \frac{d^2u}{d\theta^2} = \frac{r^2}{p^2} \cdot \frac{dp}{dr}.$$

The Perpendicular upon an Asymptote.

323. When the point of contact P passes to infinity the tangent at P becomes an asymptote, and the subtangent OT coincides with the perpendicular upon the asymptote. Hence (θ_1 denoting a value of θ for which r is infinite) the length of this perpendicular is given by the expression $-\left.\frac{d\theta}{du}\right]_{\theta_1}$, and like the polar subtangent is, when positive, to be laid off in the direction $\theta_1 - 90°$.

This expression for the perpendicular upon the asymptote is also easily derived by evaluating that given in Art. 244. Thus—

$$r \sin(\theta_1 - \theta)\Big]_{\theta_1} = \frac{\sin(\theta_1 - \theta)}{u}\Big]_{\theta_1} = -\frac{d\theta}{du}\Big]_{\theta_1}.$$

Points of Inflexion.

324. When, as in Fig. 73, the curve lies between the tangent and the pole, it is obvious that r and p will increase and decrease together; that is, $\frac{dp}{dr}$ will be positive. When on the other hand the curve lies on the other side of the tangent, $\frac{dp}{dr}$ is negative. Hence at a point of inflexion $\frac{dp}{dr}$ must change sign.

Now, since p is always positive, it follows from the equation deduced in Art. 322 that the sign of this expression is the same as that of

$$u + \frac{d^2u}{d\theta^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

hence *at a point of inflexion this expression must change sign.*

325. As an illustration, let us determine the point of inflexion of the curve traced in Art. 245; viz.,

$$r = \frac{a\theta^2}{\theta^2 - 1}.$$

In this case $$u = \frac{1}{a}\left(1 - \theta^{-2}\right);$$

whence $$\frac{du}{d\theta} = \frac{2}{a}\theta^{-3}, \quad \text{and} \quad \frac{d^2u}{d\theta^2} = -\frac{6}{a}\theta^{-4};$$

therefore $$u + \frac{d^2u}{d\theta^2} = \frac{1}{a}\left(1 - \theta^{-2} - 6\theta^{-4}\right)$$

$$= \frac{\theta^4 - \theta^2 - 6}{a\theta^4}.$$

Putting this expression equal to zero, the real roots are

$$\theta = \pm \sqrt{3},$$

and it is evident that, as θ passes through either of these values, the expression $u + \frac{d^2u}{d\theta^2}$ changes sign. Hence the points of inflexion are determined by

$$\theta = \pm \sqrt{3} \quad \text{and} \quad r = \frac{3a}{2}.$$

326. When a curve has a branch passing through the origin without inflexion, the corresponding value of p, which is zero, evidently constitutes a minimum value, and consequently $\frac{dp}{dr}$ and $u + \frac{d^2u}{d\theta^2}$ change sign. The latter expression in fact passes through infinity; hence the above method is inapplicable at the pole. The existence of a point of inflexion situated at the pole is however indicated by the fact that two values of r having opposite signs become zero for the same value of θ. For example, in the case of the lemniscata

$$r = \pm a \sqrt{(\cos 2\theta)},$$

both values of r become zero when $\theta = 45°$ and when $\theta = 135°$; hence this curve has two points of inflexion at the pole.

The Polar Equation of a Pedal.

327. Since the locus of R in Fig. 71, or in Fig. 73, is the pedal of the given curve with reference to O as the pedal origin (see Art. 307), p is the radius vector of the pedal, and its inclination to the initial line, which is $\phi - 90°$, is the corresponding vectorial angle. Hence, if we put

$$r_1 = p \quad \text{and} \quad \theta_1 = \phi - \tfrac{1}{2}\pi,$$

the relation between r_1 and θ_1 will constitute the equation of the pedal.

Thus in the case of the four-cusped hypocycloid the values of p and ϕ determined in Art. 316 are

$$p = a \sin\psi \cos\psi, \quad \text{and} \quad \phi = \pi - \psi;$$

whence $$\theta_1 = \tfrac{1}{2}\pi - \psi.$$

Putting r_1 for p and eliminating ψ, we have

$$r_1 = a\cos\theta_1 \sin\theta_1 = \tfrac{1}{2}a\sin 2\theta_1.$$

328. When the equation of the curve is given in terms of x and y, it is convenient to eliminate ϕ at once from the value of p given in equation (1) Art. 316. Thus, since $\phi = \theta_1 + \frac{1}{2}\pi$,

$$r_1 = x\cos\theta_1 + y\sin\theta_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which x and y are to be expressed in terms of θ_1. For example, if the given curve is the common parabola

$$y^2 = 4ax,$$

we have
$$\tan\phi = \frac{2a}{y};$$

whence
$$y = 2a\cot\phi \quad \text{and} \quad x = a\cot^2\phi,$$

or
$$y = -2a\tan\theta_1 \quad \text{and} \quad x = a\tan^2\theta_1.$$

Substituting in equation (1) we have

$$r_1 = a\frac{\sin^2\theta_1}{\cos\theta_1} - 2a\frac{\sin^2\theta_1}{\cos\theta_1} = -a\frac{\sin^2\theta_1}{\cos\theta_1},$$

which is the polar equation of a cissoid. Hence the pedal of the parabola with reference to its vertex is the cissoid.

329. When the equation of the given curve is in polar coordinates, the most convenient arrangement of formulas is that given below. By Fig. 73, Art. 320,

$$\phi = \theta + \psi, \quad \text{hence} \quad \theta_1 = \theta + \psi - \tfrac{1}{2}\pi. \quad . \quad . \quad (1)$$

If we eliminate ψ from the latter equation by means of the formula

$$\cot\psi = \frac{dr}{rd\theta}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

we shall have a relation between θ_1 and θ.

Again the value of p (Art. 321) gives

$$r_1 = r\sin\psi = \frac{r}{\sqrt{(1 + \cot^2\psi)}}, \quad . \quad . \quad . \quad . \quad (3)$$

in which $\cot\psi$ is given by equation (2), and r by the equation of the curve. Finally we eliminate θ from this result by means of the relation between θ and θ_1.

330. For example to find the pedal of the curve

$$r^m = a^m \cos m\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

we have
$$\log r = \log a + \frac{1}{m}\log\cos m\theta;$$

hence in this case equations (2) and (3) of the preceding article become

$$\cot\psi = -\tan m\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and
$$r_1 = a\frac{(\cos m\theta)^{\frac{1}{m}}}{\sec m\theta} = a(\cos m\theta)^{\frac{m+1}{m}}. \quad . \quad . \quad . \quad (3)$$

From (2) we have $\psi = m\theta + \frac{1}{2}\pi$. Now $\theta = 0$ in (1) gives $r = a$ and hence determines a real point of the curve, and, since $\sin\psi$ is always positive, we must take $\psi = \frac{1}{2}\pi$ when $\theta = 0$; therefore

$$\psi = m\theta + \tfrac{1}{2}\pi.$$

Hence from equation (1) of the preceding article

$$\theta_1 = (m + 1)\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and, eliminating θ between (3) and (4),

$$r_1 = a\cos\left(\frac{m\theta_1}{m+1}\right)^{\frac{m+1}{m}},$$

or

$$r_1^{\frac{m}{m+1}} = a^{\frac{m}{m+1}}\cos\left(\frac{m}{m+1}\theta_1\right). \quad . \quad . \quad . \quad . \quad (5)$$

This equation is of the same form as equation (1), and may be derived from it by putting $\dfrac{m}{m+1}$ in place of m.

331. The inverse of any curve in this class belongs also to the same class of curves. For, from equation (1), we have

$$r = a(\cos m\theta)^{\frac{1}{m}},$$

and, taking as the modulus a^2, we have for the inverse

$$r = a(\cos m\theta)^{-\frac{1}{m}};$$

but $\cos m\theta = \cos(-m\theta)$, hence the preceding equation is equivalent to

$$r^{-m} = a^{-m}\cos(-m\theta), \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

which may be obtained from (1) by writing $-m$ for m.

Again, since by Art. 309 the reciprocal polar is the inverse of the pedal, the reciprocal polar of (1) is the inverse of (5); its equation is therefore

$$r^{-\frac{m}{m+1}} = a^{-\frac{m}{m+1}}\cos\left(-\frac{m}{m+1}\theta\right). \quad . \quad . \quad . \quad (7)$$

Thus the pedal, the inverse and the reciprocal polar of each of the curves of the class represented by equation (1), are likewise members of that class of curves. Many well-known curves can be shown to belong to this class. See Examples 21, 22, and 23.

Examples XXXIII.

1. Prove that, in the case of the lemniscata $r^2 = a^2 \cos 2\theta$,

$$\psi = 2\theta + \tfrac{1}{2}\pi, \quad \text{and} \quad \frac{ds}{d\theta} = \frac{a^2}{r}.$$

2. Prove that the polar subtangent of the hyperbolic spiral is constant.

3. Find the subtangent and the subnormal of the spiral of Archimedes, and prove that $\dfrac{ds}{dr} = \dfrac{\sqrt{(a^2 + r^2)}}{a}$.

4. Find the subtangent of the lituus $r^2 = \dfrac{a^2}{\theta}$, and prove that the perpendicular from the origin upon the tangent is

$$\frac{2a\sqrt{\theta}}{\sqrt{(1 + 4\theta^2)}}.$$

5. Find the polar subtangent of the spiral $r\,(\varepsilon^\theta + \varepsilon^{-\theta}) = a$.

$$-\frac{a}{\varepsilon^\theta - \varepsilon^{-\theta}}.$$

6. Find the value of p in the case of the curve $r^n = a^n \sin n\theta$.

$$p = a\,(\sin n\theta)^{1+\frac{1}{n}}.$$

7. In the case of the parabola referred to the focus

$$r = \frac{2a}{1 + \cos\theta}, \text{ prove that } p^2 = ar.$$

8. In the case of the equilateral hyperbola

$$r^2 \cos 2\theta = a^2, \text{ prove that } p = \frac{a^2}{r}.$$

9. In the case of the lemniscata

$$r^2 = a^2 \cos 2\theta, \text{ prove that } p = \frac{r^3}{a^2}.$$

10. In the case of the ellipse $r = \dfrac{a(1 - e^2)}{1 - e\cos\theta}$, the pole being at the focus, determine p.

$$p = \frac{a(1 - e^2)}{\sqrt{(1 - 2e\cos\theta + e^2)}}.$$

11. In the case of the cardioid

$$r = a(1 + \cos\theta), \text{ prove that } r^3 = 2ap^2.$$

12. Determine the asymptotes of the hyperbola by the method of Art. 323, the polar equation being

$$r = \frac{a(e^2 - 1)}{1 - e\cos\theta}.$$

$$\theta_1 = \pm \sec^{-1} e, \quad p = \mp a\sqrt{(e^2 - 1)}.$$

13. Prove that the condition which determines points of inflexion in polar coordinates; namely, that $u + \dfrac{d^2u}{d\theta^2}$ shall change sign, is equivalent to the condition that $r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}$ shall change sign.

14. Determine the points of inflexion of the lituus $r^2\theta = a^2$.

$$\theta = \tfrac{1}{2}, \text{ and } r = \pm a\sqrt{2}.$$

15. Show that the curve $r\theta^m = a$ has points of inflexion determined by $\theta = \sqrt{[m(1-m)]}$.

16. Show that the curve $r\theta \sin\theta = a$ has a point of inflexion at which $r = \frac{2a}{\pi}$.

17. Show that the curve $r = b\theta^n$ has a point of inflexion determined by $r = b\,[-n\,(n+1)]^{\frac{n}{2}}$.

18. Show that, if the curve $r = \frac{f(\theta)}{F(\theta)}$ has an asymptote whose inclination to the initial line is θ_1, the perpendicular on it will be

$$-\frac{f(\theta_1)}{F'(\theta_1)}.$$

19. Show that the curve $r\,(2\theta - 1) = 2a\theta$ has an asymptote determined by $\theta = \frac{1}{2}$, $p = -\frac{1}{2}a$, and a point of inflexion determined by the real root of the equation $2\theta^3 - \theta^2 - 2 = 0$.

20. Determine the asymptotes to the curve $r = \frac{a}{\cos k\,(\theta - \alpha)}$, for the branch which passes through the point (a, α).

$$\theta = \alpha \pm \frac{\pi}{2k},\ p = \pm\frac{a}{k}.$$

This curve is one of Cotes' Spirals. See *Tait and Steele's Dynamics*, fourth edition, London, 1878, p. 150.

21. Show that if $m = -\frac{1}{2}$ the general equation of Art. 330 represents a parabola referred to its focus, and determine its pedal and its reciprocal.

The pedal is the straight line $r\cos\theta = a$;
the reciprocal is the circle $r = a\cos\theta$.

22. Show that if $m = \frac{1}{2}$ in the general equation of Art. 330 the curve is the cardioid whose axis is a, the pole being at the cusp; and

thence determine the equations of the pedal and of the reciprocal to this curve.

Pedal $r = a\cos^2\frac{1}{2}\theta$;
reciprocal $r = a\sec^2\frac{1}{2}\theta$.

23. Determine the curves, belonging to the class discussed in Art. 330, of which the inverse is identical with the pedal, and consequently the reciprocal identical with the original curve.

$m = 0$, and $m = -2$.

The former gives the circle and the latter the equilateral hyperbola each referred to the centre. The inverse and pedal of the latter curve is the lemniscata.

24. Show that the nth pedal of the curve

$$r^m = a^m \cos m\theta$$

is determined by putting $\dfrac{m}{mn+1}$ for m in the general equation; and that when n is negative, the same formula gives the successive negative pedals.

25. Find the pedal and the reciprocal polar of the equiangular spiral $r = a\varepsilon^{n\theta}$.

Each of these curves is a spiral equal to the original spiral, but in a different position.

26. Find the reciprocal polar of the curve $r = a\sec^2\frac{1}{2}\theta$.

$r_1^{\frac{1}{2}} = a^{\frac{1}{2}}\cos\frac{1}{2}\theta_1$.

27. Find the pedal of the circle $(x-b)^2 + y^2 = a^2$.

The limaçon $r_1 = b\cos\theta_1 \pm a$.

28. Find the polar equation of the pedal of the ellipse, the centre being the pedal origin.

$r_1^2 = a^2\cos^2\theta_1 + b^2\sin^2\theta_1$.

29. Prove that, in the case of the *three-cusped hypocycloid* in which $a = 3b$,

$$\tan\phi = -\tan\tfrac{1}{3}\psi;$$

and thence find the pedal of the curve.

$$r_1 = -b\cos 3\theta_1.$$

30. Find the pedal of the epicycloid. See Art. 293.

$$r_1 = (a + 2b)\sin\left[\frac{a(\theta_1 + \frac{1}{2}\pi)}{a + 2b}\right].$$

XXXIV.

Curvature.

332. If, while a point P moves along a given curve at the rate $\dfrac{ds}{dt}$, it be regarded as carrying with it the tangent and normal lines, each of these lines will rotate about the moving point P at the angular rate $\dfrac{d\phi}{dt}$, ϕ denoting the inclination of the tangent line to the axis of x.

The point P is always moving in a direction perpendicular to the normal with the velocity $\dfrac{ds}{dt}$. Let us consider the motion of a point A on the normal at a given distance k from P on the concave side of the arc.

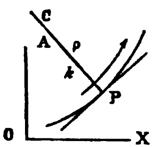


FIG. 74.

While this point is carried forward by the motion of P with the velocity $\dfrac{ds}{dt}$ in a direction perpendicular to the normal, it is at the same time carried backward, by the rotation of this line about P, with the

velocity $\frac{kd\phi}{dt}$; since this is the velocity with which A would move if the point P occupied a fixed position in the plane; and the *direction* of this motion is evidently directly opposite to that of P. Hence the actual velocity of A will be

$$\frac{ds}{dt} - k\frac{d\phi}{dt},$$

in a direction parallel to the tangent at P.

Let ρ denote the value of k which reduces this expression to zero, and let C (Fig. 74) be the corresponding position of A: then,

$$\frac{ds}{dt} - \rho\frac{d\phi}{dt} = 0;$$

whence $$PC = \rho = \frac{ds}{d\phi}. \quad \quad (1)$$

333. The value of ρ determined by this equation is, in general, variable; for, if the point P move along the curve with a given linear velocity $\frac{ds}{dt}$, the angular velocity $\frac{d\phi}{dt}$ will generally be variable. If however we suppose the angular velocity $\frac{d\phi}{dt}$ to become constant, at the instant when P passes a given position on the curve, $\frac{ds}{d\phi}$, the value of ρ, will likewise become constant, and C will remain stationary. When this hypothesis is made, the curvature of the path of P becomes constant, for P describes a circle whose centre is C, and whose radius is ρ. Hence this circle is called *the circle of curvature* corresponding to the given position of P; C is accordingly called *the centre of curvature*, and ρ is called *the radius of curvature*.

The Direction of the Radius of Curvature.

334. If in Fig. 74 the arrow indicates the positive direction of s; the case represented is that in which ϕ and s increase together, and therefore the value of ρ as determined by equation (1), Art. 332, is positive. Hence it is evident that when ρ is positive its direction from P is that of PC in Fig. 74; namely, $\phi + 90°$. In other words, to a person looking along the curve in the positive direction of ds, ρ, when positive, is laid off on *the left-hand side of the curve.*

For example, let the curve be the four-cusped hypocycloid,

$$x = a\cos^3\psi, \qquad y = a\sin^3\psi.$$

It was shown in Art. 316 that for this curve

$$ds = 3a\sin\psi\cos\psi\, d\psi, \quad \text{and} \quad \phi = \pi - \psi;$$

hence

$$d\phi = -\, d\psi,$$

and

$$\rho = \frac{ds}{d\phi} = -\, 3a\sin\psi\cos\psi. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

When ψ is in the first quadrant ρ is negative; its direction is therefore $\phi - \frac{1}{2}\pi = \frac{1}{2}\pi - \psi$, which is in the first quadrant. When ψ is in the second quadrant ρ is positive and its direction is $\phi + \frac{1}{2}\pi = \frac{3}{2}\pi - \psi$, which is in the second quadrant.

The Radius of Curvature in Rectangular Coordinates.

335. To express ρ in terms of derivatives with reference to x, we have

$$\phi = \tan^{-1}\frac{dy}{dx}, \quad \text{and} \quad \frac{ds}{dx} = \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]};$$

hence
$$\frac{d\phi}{dx} = \frac{\dfrac{d^2y}{dx^2}}{1 + \left(\dfrac{dy}{dx}\right)^2},$$

and
$$\rho = \frac{\dfrac{ds}{dx}}{\dfrac{d\phi}{dx}} = \frac{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Since $\dfrac{ds}{dx}$ is assumed to be positive, ϕ should be so taken as to cause x to increase with s, and it must be remembered that the dirction of ρ is $\phi + 90°$ when ρ is positive, in accordance with the remark in the preceding article.

336. To illustrate the application of the above formula, we find the radius of curvature of the ellipse

$$y = \pm\frac{b}{a}\sqrt{(a^2 - x^2)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Differentiating,
$$\frac{dy}{dx} = \mp\frac{bx}{a\sqrt{(a^2 - x^2)}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and
$$\frac{d^2y}{dx^2} = \mp\frac{ab}{(a^2 - x^2)^{\frac{3}{2}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Putting $b = a\sqrt{(1 - e^2)}$ we obtain

$$1 + \left(\frac{dy}{dx}\right)^2 = \frac{a^2 - e^2x^2}{a^2 - x^2};$$

whence, substituting in equation (1) of the preceding article,

$$\rho = \mp \frac{(a^2 - e^2x^2)^{\frac{3}{2}}}{a^2\sqrt{(1-e^2)}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

337. When this result is applied to a meridian of the earth regarded as an ellipse, it is usual to express ρ in terms of the latitude l.

The definition of l gives

$$l = \phi - \tfrac{1}{2}\pi;$$

hence, from equation (2),

$$\tan^2 l = \frac{a^2(a^2 - x^2)}{b^2x^2} = \frac{a^2 - x^2}{(1-e^2)x^2};$$

therefore $$x^2 = \frac{a^2}{1 + (1-e^2)\tan^2 l} = \frac{a^2}{\sec^2 l - e^2\tan^2 l},$$

and $$a^2 - e^2x^2 = \frac{a^2(\sec^2 l - e^2\sec^2 l)}{\sec^2 l - e^2\tan^2 l} = \frac{a^2(1-e^2)}{1 - e^2\sin^2 l}.$$

Whence, substituting in equation (4),

$$\rho = \frac{a(1-e^2)}{(1 - e^2\sin^2 l)^{\frac{3}{2}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The Radius of Curvature at the Origin when the Tangent and Normal are the Coordinate Axes.

338. When the tangent and normal are the coordinate axes, the formula deduced below affords an easy method of determining the radius of curvature at the origin.

If the axis of x is a tangent to the curve at the origin, we obviously have

$$ds = dx,$$

and the value of ρ given in equation (1) Art. 335 becomes

$$\rho_0 = \frac{1}{\left.\frac{d^2y}{dx^2}\right]_0}. \qquad (1)$$

But when the axis of x is a tangent at the origin, we have

$$\left.\frac{dy}{dx}\right]_0 = \left.\frac{y}{x}\right]_0 = 0;$$

therefore $\left.\frac{y}{x^2}\right]_0$ assumes the indeterminate form $\frac{0}{0}$; hence, evaluating,

$$\left.\frac{y}{x^2}\right]_0 = \left.\frac{\frac{dy}{dx}}{2x}\right]_0 = \frac{1}{2}\left.\frac{d^2y}{dx^2}\right]_0, \qquad (2)$$

therefore, by equation (1),

$$\rho_0 = \left.\frac{x^2}{2y}\right]_0. \qquad (3)$$

It is to be noticed, when this formula is applied, that ϕ is zero because ds was assumed equal to dx; and hence that, by Art. 334, the direction of ρ when positive is 90°; in other words, the centre of curvature is *above the origin* when ρ is *positive.**

Again, when the axis of y is a tangent at the origin, we have

$$\rho_0 = \left.\frac{y^2}{2x}\right]_0, \qquad (4)$$

* When the equation of the curve is given in polar coordinates, the initial line being tangent at the pole, we have, for the value of ρ at this point,

$$\rho_0 = \left.\frac{x^2}{2y}\right]_0 = \left.\frac{r^2\cos^2\theta}{2r\sin\theta}\right]_{\theta=0} = \left.\frac{r}{2\theta}\right]_0.$$

the centre of curvature being *on the right of the origin* when ρ is *positive.*

The method of determining curvature used by Sir Isaac Newton in *The Principia* is equivalent to the employment of the above formulas.

339. As an example let us take the equation of a conic,

$$y^2 + (1 - e^2)x^2 - 2a(1 - e^2)x = 0.$$

This curve is tangent to the axis of y, since the equation of the tangent at the origin is $x = 0$. See Art. 218.

Dividing the equation by $2x$, we have

$$\frac{y^2}{2x} + (1 - e^2)\frac{x}{2} - a(1 - e^2) = 0,$$

and putting $x = 0$ we derive by equation (4)

$$\rho_0 = a(1 - e^2).$$

In applying these formulas when the origin is a node the method illustrated below is convenient. Let the equation of the curve be

$$x^5 + ax^2y^2 - ax^3y - 2a^2xy^2 + a^2y^3 = 0,$$

y^2 being a factor of the terms of lowest degree, we employ formula (3), whence

$$y = \frac{x^2}{2\rho};$$

substituting in the equation of the curve, we obtain

$$x^5 - a\frac{x^6}{4\rho^2} - a\frac{x^5}{2\rho} - a^2\frac{x^5}{2\rho^2} + a^2\frac{x^6}{8\rho^3} = 0;$$

dividing by the lowest power of x that appears, and putting $x = 0$ in the resulting equation, we have

$$2\rho^2 - a\rho - a^2 = 0;$$

whence $$\rho = a \quad \text{and} \quad \rho = -\tfrac{1}{2}a.$$

It follows that the curve has two branches touching the axis of x at the origin, one branch being above the axis and the other below it.

Expressions for ρ in which x is not the Independent Variable.

340. To express ρ in terms of derivatives with reference to y, we have

$$\phi = \cot^{-1}\frac{dx}{dy}, \quad \text{and} \quad \frac{ds}{dy} = \sqrt{\left[1 + \left(\frac{dx}{dy}\right)^2\right]};$$

whence $$\frac{d\phi}{dy} = -\frac{\dfrac{d^2x}{dy^2}}{1 + \left(\dfrac{dx}{dy}\right)^2}, \quad \text{and} \quad \rho = -\frac{\left[1 + \left(\dfrac{dx}{dy}\right)^2\right]^{\frac{3}{2}}}{\dfrac{d^2x}{dy^2}}.$$

In this case ds and dy were assumed to have the same sign, hence ϕ must be taken so as to cause y to increase.

341. When x and y are expressed in terms of a third variable we employ the formula deduced below.

Differentiating

$$\phi = \tan^{-1}\frac{dy}{dx},$$

both dx and dy being regarded as variable, we have

$$d\phi = \frac{\dfrac{dx\,d^2y - dy\,d^2x}{dx^2}}{1 + \left(\dfrac{dy}{dx}\right)^2} = \frac{dx\,d^2y - dy\,d^2x}{dx^2 + dy^2};$$

whence
$$\rho = \frac{ds}{d\phi} = \frac{(dx^2 + dy^2)^{\frac{3}{2}}}{dx\,d^2y - dy\,d^2x}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

342. If, to determine $d\phi$ we employ the relation

$$\sin\phi = \frac{dy}{ds},$$

we shall obtain another formula for determining ρ. Thus, differentiating, we have

$$\cos\phi\,d\phi = \frac{ds\,d^2y - dy\,d^2s}{ds^2},$$

and, since
$$\cos\phi = \frac{dx}{ds},$$

$$d\phi = \frac{ds\,d^2y - dy\,d^2s}{dx\,ds};$$

substituting this value of $d\phi$, we obtain

$$\rho = \frac{dx\,ds^2}{ds\,d^2y - dy\,d^2s}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Introducing derivatives with reference to s, we have, since $\dfrac{d^2s}{ds^2} = 0$,

$$\rho = \frac{\dfrac{dx}{ds}}{\dfrac{d^2y}{ds^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

343. In like manner, by employing

$$\cos\phi = \frac{dx}{ds}$$

to eliminate $d\phi$, we derive

$$\rho = -\frac{dy\,ds^2}{ds\,d^2x - dx\,d^2s}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and, in terms of derivatives with reference to s,

$$\rho = -\frac{\dfrac{dy}{ds}}{\dfrac{d^2x}{ds^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

344. The expressions for ρ deduced below are employed in deriving certain results in Dynamics.

We have

$$ds\cos\phi = dx \qquad \text{and} \qquad ds\sin\phi = dy;$$

whence, differentiating,

$$d^2s\cos\phi - ds\sin\phi\,d\phi = d^2x,$$

and

$$d^2s\sin\phi + ds\cos\phi\,d\phi = d^2y.$$

Squaring and adding, we obtain

$$(d^2s)^2 + ds^2\,d\phi^2 = (d^2x)^2 + (d^2y)^2,$$

whence

$$d\phi^2 = \frac{(d^2x)^2 + (d^2y)^2 - (d^2s)^2}{ds^2};$$

substituting $\dfrac{ds}{\rho}$ for $d\phi$, and solving for ρ, we have

$$\rho = \frac{ds^2}{[(d^2x)^2 + (d^2y)^2 - (d^2s)^2]^{\frac{1}{2}}} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

By making s the independent variable, we obtain the derivative expression

$$\frac{1}{\rho} = \left[\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2\right]^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The Radius of Curvature in Polar Coordinates.

345. To obtain an expression for ρ in terms of r and θ, we have

$$\phi = \psi + \theta, \quad \text{whence} \quad \rho = \frac{ds}{d\phi} = \frac{ds}{d\theta + d\psi}, \quad . \quad . \quad (1)$$

and

$$\tan\psi = \frac{rd\theta}{dr}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In differentiating equation (2) to obtain an expression for $d\psi$, $d\theta$ may be regarded as constant; since the result is to be expressed in derivatives with reference to θ.

Hence
$$\sec^2\psi\, d\psi = \frac{dr^2 - rd^2r}{dr^2}\, d\theta,$$

and, since $\sec\psi = \dfrac{ds}{dr}$,

$$d\psi = \frac{dr^2 - rd^2r}{ds^2}\, d\theta.$$

Hence
$$d\theta + d\psi = \frac{dr^2 + ds^2 - rd^2r}{ds^2}\, d\theta,$$

and, substituting in (1), we obtain

$$\rho = \frac{ds^3}{(dr^2 + ds^2 - rd^2r)\, d\theta}.$$

but
$$ds^2 = dr^2 + r^2d\theta^2,$$

therefore
$$\rho = \frac{(dr^2 + r^2d\theta^2)^{\frac{3}{2}}}{(2dr^2 + r^2d\theta^2 - rd^2r)\,d\theta},$$

or
$$\rho = \frac{\left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right]^{\frac{3}{2}}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}. \quad . \quad . \quad . \quad . \quad (3)$$

346. To obtain ρ in terms of u, we eliminate r from the above equation thus:

$$r = \frac{1}{u}, \qquad \text{then} \qquad \frac{dr}{d\theta} = -\frac{1}{u^2}\cdot\frac{du}{d\theta},$$

and
$$\frac{d^2r}{d\theta^2} = \frac{2}{u^3}\left(\frac{du}{d\theta}\right)^2 - \frac{1}{u^2}\cdot\frac{d^2u}{d\theta^2}.$$

On substituting these values, we obtain

$$\rho = \frac{\left\{u^2 + \left(\frac{du}{d\theta}\right)^2\right\}^{\frac{3}{2}}}{u^3\left(u + \frac{d^2u}{d\theta^2}\right)}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

347. In the case of a polar curve, when there exists a simple relation between ψ and θ, the expression

$$\rho = \frac{ds}{d\phi}$$

may usually be employed with advantage. Thus the polar equation of the cardioid (Art. 271) is

$$r = 4a\sin^2\tfrac{1}{2}\theta;$$

whence [see equation (3) Art. 318]

$$\cot\psi = \frac{dr}{rd\theta} = \cot\tfrac{1}{3}\theta,$$

$$\psi = \tfrac{1}{3}\theta \qquad \text{and} \qquad \phi = \theta + \psi = \tfrac{4}{3}\theta.$$

Therefore $$\rho = \frac{ds}{d\phi} = \tfrac{3}{4}\frac{ds}{d\theta} = \tfrac{3}{4}\, r \operatorname{cosec}\psi = \tfrac{3}{4}\, a \sin \tfrac{1}{3}\theta.$$

Relations between ρ, p, and τ.

348. In Fig. 75, if we denote OR by p and PR by τ, we shall have

$$p = r\sin\psi, \quad \text{and} \quad \tau = r\cos\psi \quad . \quad . \quad (1)$$

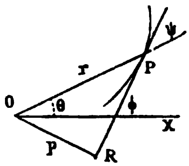


FIG. 75.

Now let P move along the curve at the rate $\dfrac{ds}{dt}$, then the tangent PR will rotate about P at the angular rate $\dfrac{d\phi}{dt}$, and OR will rotate about O at the same rate, since these lines are always at right angles to each other. The motion of the point R may be resolved into two motions; one in the direction OR, and the other in the direction RP. Since the velocity of P in the direction OR is zero, the component of the velocity of R in this direction is $\tau\dfrac{d\phi}{dt}$, while the component in the direction RP is $p\dfrac{d\phi}{dt}$. The first of these components is the rate of p, since O is a fixed point; therefore

$$\frac{dp}{dt} = \tau\frac{d\phi}{dt} \qquad \text{whence} \qquad \tau = \frac{dp}{d\phi} \quad . \quad . \quad . \quad (2)$$

The rate of τ is the difference between the velocity of P in

the direction RP, and the component velocity of R in the same direction; therefore

$$\frac{d\tau}{dt} = \frac{ds}{dt} - p\frac{d\phi}{dt}, \quad \text{or} \quad d\tau = ds - pd\phi. \quad . \quad . \quad (3)$$

349. By comparing the expressions for τ in equations (1) and (2), and putting for $\cos\psi$ its value $\frac{dr}{ds}$, we obtain

$$\frac{dp}{d\phi} = r\frac{dr}{ds};$$

whence

$$\rho = \frac{ds}{d\phi} = \frac{rdr}{dp}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

An expression for ρ may also be derived from equation (3); thus,

$$\rho = \frac{ds}{d\phi} = p + \frac{d\tau}{d\phi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

or, by equation (2),

$$\rho = p + \frac{d^2p}{d\phi^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

350. The point R in Fig. 75 describes the pedal of the given curve, the polar coordinates of the pedal (Art. 327) being

$$\theta_1 = \phi - \tfrac{1}{2}\pi \quad \text{and} \quad r_1 = p.$$

The ratio of the component velocities of R determines the direction in which R moves; hence, denoting by ψ_1 the value of ψ for this curve, we have

$$\tan\psi_1 = \frac{p}{\tau} = \tan\psi.$$

Hence the angle between a curve and its radius vector at any point is equal to the angle between the pedal and its radius vector at the corresponding point. It is thence easily shown that *the normal to the pedal bisects OP.*

351. From equation (2) Art. 348, we derive

$$p\tau = p\frac{dp}{d\phi} = \frac{d(p^2)}{2d\phi};$$

this expression enables us to obtain the value of the product $p\tau$ when p is expressed in terms of the angle θ_1, its inclination to the axis of x. Thus, in the case of the ellipse referred to its centre, we have

$$p^2 = a^2\cos^2\theta_1 + b^2\sin^2\theta_1 \,^*;$$

since $d\theta_1 = d\phi$ we obtain, by differentiating,

$$p\tau = -(a^2 - b^2)\cos\theta_1\sin\theta_1.$$

352. *The chord of curvature through the origin* is that chord of the circle of curvature, corresponding to any given point P of a curve, which passes through P and through the origin. Denoting this chord by $2c$, we have

$$2c = 2\rho\sin\phi = 2\rho\frac{p}{r};$$

whence, putting $\frac{dr}{dp}$ for $\frac{\rho}{r}$ [see equation (4) Art. 349],

$$2c = 2p\frac{dr}{dp}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

* This equation, which may be found in works on Conic Sections, is equivalent to the polar equation of the pedal to the ellipse, when the centre is the pedal origin. (See Ex. XXXIII, 28.)

To derive an expression for $2c$ in terms of u, we have [equation (2), Art. 321]

$$p = \left[u^2 + \left(\frac{du}{d\theta}\right)^2\right]^{-\frac{1}{2}}.$$

Whence, taking logarithmic differentials,

$$\frac{dp}{p} = -\frac{\left(u + \dfrac{d^2u}{d\theta^2}\right)du}{u^2 + \left(\dfrac{du}{d\theta}\right)^2}.$$

Substituting in equation (1), and putting for dr its value $-\dfrac{du}{u^2}$, we obtain

$$2c = 2\frac{u^2 + \left(\dfrac{du}{d\theta}\right)^2}{u^3\left(u + \dfrac{d^2u}{d\theta^2}\right)}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Examples XXXIV.

1. Derive the expression

$$\frac{1}{\rho} = \left[\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2\right]^{\frac{1}{2}}$$

from equations (2) Art. 342, and (2) Art. 343, by means of the relation

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1.$$

2. Find the radius of curvature of the *cycloid*

$$x = a(\psi - \sin\psi), \qquad y = a(1 - \cos\psi).$$

Prove that $\phi = \frac{1}{2}(\pi - \psi)$, *and use* $\rho = \frac{ds}{d\phi}$.

$$\rho = -2\sqrt{(2ay)}.$$

3. Find the radius of curvature of the *involute of the circle*, Art. 301,

$$x = a\cos\psi + a\psi\sin\psi, \quad y = a\sin\psi - a\psi\cos\psi.$$

$$\rho = a\psi.$$

4. Find the radius of curvature of the *parabola* $y^2 = 4ax$.

$$\rho = \mp 2\,\frac{(a+x)^{\frac{3}{2}}}{\sqrt{\cancel{a}}}.$$

x

5. Find the radius of curvature of the *catenary*

$$y = \frac{c}{2}\left(\varepsilon^{\frac{x}{c}} + \varepsilon^{-\frac{x}{c}}\right),$$

and show that its numerical value equals that of the normal at the same point.

$$\rho = \frac{y^2}{c}. \quad \checkmark$$

6. Find the radius of curvature of the *semi-cubical parabola*

$$ay^2 = x^3.$$

$$\rho = \frac{(4a + 9x)^{\frac{3}{2}}x^{\frac{1}{2}}}{6a}.$$

7. Find the radius of curvature of the *logarithmic curve*

$$y = a\varepsilon^{\frac{x}{c}}.$$

$$\rho = \frac{[c^2 + y^2]^{\frac{3}{2}}}{cy}. \quad \checkmark$$

8. Find the radius of curvature of the *tractrix*

$$\frac{dx}{dy} = -\frac{(a^2 - y^2)^{\frac{1}{2}}}{y}.$$

$$\rho = \frac{a}{y}(a^2 - y^2)^{\frac{1}{2}}.$$

9. Find the radius of curvature of the *rectangular hyperbola*

$$xy = m^2.$$

$$\rho = \frac{(x^2 + y^2)^{\frac{3}{2}}}{2m^2}.$$

10. Find the radius of curvature of the *hyperbola*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

$$\rho = \mp \frac{(e^2x^2 - a^2)^{\frac{3}{2}}}{ab}.$$

11. Find the radius of curvature of the *parabola of the* n*th degree*

$$a^{n-1}y = x^n.$$

$$\rho = \frac{(x^2 + n^2y^2)^{\frac{3}{2}}}{n(n-1)xy}.$$

12. Find the radius of curvature of the *cissoid*

$$y = \frac{x^{\frac{3}{2}}}{(2a - x)^{\frac{1}{2}}}.$$

$$\rho = \frac{a\sqrt{x}\,(8a - 3x)^{\frac{3}{2}}}{3(2a - x)^2}.$$

13. Given the curve $y^4 + x^4 + a(x^2 + y^2) = a^2y$.
Find the value of ρ at the origin. See Art. 338.

$$\rho_0 = \frac{a}{2}.$$

14. Given the curve $a^2y = bx^3 + cx^2y$.
Determine the value of ρ_0.

$$\rho_0 = \infty.$$

15. Given the parabola $a^{n-1}y = x^n$.
Determine the value of ρ_0.

For $n < 2$ we have $\rho_0 = 0$;
" $n = 2$ " $\rho_0 = \frac{a}{2}$;
" $n > 2$ " $\rho_0 = \infty$.

16. Given the curve $ax^3 - 2b^2xy + cy^3 = x^4 + y^4$.
Determine the values of ρ_0.

For the branch tangent to the axis of x, $\rho_0 = \frac{b^2}{a}$;

for the branch tangent to the axis of y, $\rho_0 = \frac{b^2}{c}$.

17. Given the curve $x^4 - 3axy^2 + 2ay^3 = 0$.
Find the value of the radius of curvature at the cusp. See Fig. 20, Art. 215.

$$\rho_0 = 0.$$

18. Given the curve $x^4 - ax^2y + axy^2 + \frac{1}{4}a^2y^2 = 0$.
Determine the value of ρ_0. See Ex. XXVIII, 24.

$$\rho_0 = \tfrac{1}{4}a.$$

19. Show by means of the expression for ρ_0 that the curve

$$x^4 - ax^2y - axy^2 + a^2y^2 = 0$$

has an isolated point at the origin. See Ex. XXVIII, 23.

20. Find the radius of curvature at the origin, the equation of the curve being [see Ex. XXVIII, 25]

$$x^4 - \tfrac{5}{2}ax^2y - axy^2 + a^2y^2 = 0.$$

$\rho_0 = a$, and $\rho_0 = \frac{1}{4}a$.

21. Given the curve $x^5 - 4ay^4 + 2ax^3y + a^2xy^2 = 0$.
Find the values of ρ_0. See Ex. XXVIII, 26.

$\rho_0 = -\frac{1}{2}a$, at the cusp;
$\rho_0 = \frac{1}{8}a$, for the other branch.

22. Given the curve $a(y^2 - x^2) = x^3$.
Determine the values of ρ_0.
Turning the axes forward 45°, *the equation of the curve becomes*

$$4\sqrt{2}\cdot axy = (x-y)^3.$$

$\rho_0 = 2a\sqrt{2}$, and $\rho_0 = -2a\sqrt{2}$.

23. If $\dfrac{dy}{dx} = \dfrac{a}{s}$ at every point of a curve, find the value of ρ.

Solution:—

$$\phi = \tan^{-1}\frac{a}{s}, \quad \text{whence} \quad \frac{d\phi}{ds} = -\frac{\dfrac{a}{s^2}}{1 + \dfrac{a^2}{s^2}};$$

therefore
$$\rho = -\frac{a^2 + s^2}{a}.$$

24. Find the radius of curvature of the parabola

$$\sqrt{x} + \sqrt{y} = 2\sqrt{a}.$$

$$\rho = \frac{(x+y)^{\frac{3}{2}}}{\sqrt{a}}.$$

25. Find the radius of curvature of the *cubical parabola*

$$a^2y = x^3.$$

$$\rho = \frac{(a^4 + 9x^4)^{\frac{3}{2}}}{6a^4x}$$

26. Find the radius of curvature of the *prolate cycloid*

$$x = a\psi - b\sin\psi, \qquad y = a - b\cos\psi.$$

$$\rho = \frac{(a^2 + b^2 - 2ab\cos\psi)^{\frac{3}{2}}}{b(a\cos\psi - b)}.$$

27. Find the radius of curvature of the *three-cusped hypocycloid*

$$x = b\,(2\cos\psi + \cos 2\psi), \qquad y = b\,(2\sin\psi - \sin 2\psi).$$

$$\rho = -\,8b\sin\tfrac{3}{2}\psi.$$

28. Find the radius of curvature of the *epicycloid*

$$x = (a+b)\cos\psi - b\cos\frac{a+b}{b}\psi, \qquad y=(a+b)\sin\psi - b\sin\frac{a+b}{b}\psi.$$

$$\rho = \frac{4b\,(a+b)}{a+2b}\sin\frac{a\psi}{2b}.$$

29. Find the radius of curvature of the curve

$$x = 2c\sin 2\psi\cos^2\psi, \qquad y = 2c\cos 2\psi\sin^2\psi.$$

$$\rho = 4c\cos 3\psi.$$

30. Find a general expression for the radius of curvature of any conic referred to its focus, the equation being

$$r = \frac{a\,(1-e^2)}{1-e\cos\theta}.$$

$$\rho = \frac{a\,(1-e^2)\,(1-2e\cos\theta+e^2)^{\frac{3}{2}}}{(1-e\cos\theta)^3}.$$

31. Find the radius of curvature of the *trisectrix*

$$r = 2a\cos\theta - a.$$

$$\rho = \frac{a\,(5-4\cos\theta)^{\frac{3}{2}}}{9-6\cos\theta}.$$

32. Find the radius of curvature of the *lemniscata*

$$r^2 = a^2\cos 2\theta.$$

See Art. 347.

$$\rho = \frac{a^2}{3r}.$$

33. Find the radius of curvature of the *parabola* whose equation is $r\cos^2\frac{1}{2}\theta = a$, the focus being the pole.

$$\rho = \frac{2r^{\frac{3}{2}}}{\sqrt{a}}.$$

34. Find the radius of curvature of the *equilateral hyperbola*

$$r^2\cos 2\theta = a^2.$$

$$\rho = -\frac{r^3}{a^2}.$$

35. Find the radius of curvature of the *equiangular spiral*

$$r = a\epsilon^{n\theta}.$$

$$\rho = r\sqrt{(1+n^2)}.$$

36. Find the radius of curvature of the *lituus*

$$r^2\theta = a^2.$$

$$\rho = \frac{r(4a^4 + r^4)^{\frac{3}{2}}}{2a^2(4a^4 - r^4)}.$$

37. Given $r^m = a^m \cos m\theta$, prove that $r^{m+1} = a^m p$, and thence derive the value of ρ by means of equation (4) Art. 349.

$$\rho = \frac{r^2}{(m+1)p}.$$

38. Show that the chord of curvature through the origin, in the case of the *cardioid* $r = 2a(1-\cos\theta)$, is $\frac{4r}{3}$.

XXXV.

Evolutes.

353. *The evolute of a curve is the locus of its centre of curvature.*

Since, when the point P, Fig. 76, moves along the given curve, the motion of the centre of curvature has no component perpendicular to the normal PC (see Art. 332), this line is tangent at C to the curve described by the point C; consequently the evolute may be defined as *the curve whose tangents are the normals to the given curve.*

Now, since the point P moves in a direction perpendicular to PC while C has no motion except in the direction PC; if we regard P as a fixed origin on the line PC, the rate of the motion of C along this line will be identical with its rate along the arc of the evolute. Hence PC may be regarded as a tangent line *rolling upon the evolute*, while the fixed point P of this tangent describes the original curve.

A curve generated by any fixed point on a rolling tangent is called an *involute.* Thus the original curve is one of the involutes to its own evolute, and may be traced by the extremity of a cord which is being wound upon the evolute or unwound from it.

354. We proceed to the methods of deriving the equation of the evolute.

To obtain expressions for the coordinates (x', y') of the centre of curvature C (Fig. 76) in terms of x and y, we project the line ρ on the coordinate axes. The inclination of PC to the axis of x is $\phi + 90°$; hence the projections are

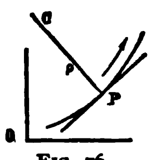


Fig. 76.

$$x' - x = -\rho \sin\phi = -\rho\frac{dy}{ds}, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and
$$y' - y = \rho\cos\phi = \rho\frac{dx}{ds}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Substituting for ρ in the above equations the value of ρ given in equation (1), Art. 335, we have

$$x' - x = -\frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} \cdot \frac{dy}{dx}, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$y' - y = \frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From these equations y and its derivatives must be eliminated by means of the equation of the given curve. The relation between x' and y' found by subsequently eliminating x from the two equations is the equation of the evolute. It should be remarked however that the latter elimination is frequently impracticable.*

355. As an illustration, let it be required to find the evolute of the common parabola

$$y = 2a^{\frac{1}{2}}x^{\frac{1}{2}};$$

whence
$$\frac{dy}{dx} = \left(\frac{a}{x}\right)^{\frac{1}{2}},$$

* Another method of obtaining the equation of the evolute will be found in Section XXXVII.

and
$$\frac{d^2y}{dx^2} = -\frac{a^{\frac{1}{2}}}{2x^{\frac{3}{2}}}$$

By substituting in formulas (3) and (4), Art. 354, these values of y, $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$, we obtain

$$x' = 2a + 3x, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and
$$y' = -\frac{2x^{\frac{3}{2}}}{a^{\frac{1}{2}}}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

eliminating x, we have

$$27ay'^2 = 4(x' - 2a)^3,$$

the equation of the evolute, which is, therefore, a *semi-cubical parabola* having its cusp at the point $(2a, 0)$.

356. When it is more convenient to regard x as a function of y, we employ the equations

$$x' - x = \frac{1 + \left(\frac{dx}{dy}\right)^2}{\frac{d^2x}{dy^2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and
$$y' - y = -\frac{1 + \left(\frac{dx}{dy}\right)^2}{\frac{d^2x}{dy^2}} \cdot \frac{dx}{dy}. \quad . \quad . \quad . \quad . \quad (2)$$

These equations are obtained by substituting in equations (1) and (2) of Art. 354 the value of ρ derived in Art. 340.

It is sometimes advantageous to use one of these formulas

in connection with one of the formulas of Art. 354. For example, to find the evolute of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1;$$

we have $$y = \pm \frac{b}{a} \sqrt{(x^2 - a^2)},$$

whence $$\frac{dy}{dx} = \pm \frac{b}{a} \frac{x}{\sqrt{(x^2 - a^2)}},$$

and $$\frac{d^2y}{dx^2} = \mp \frac{ab}{(x^2 - a^2)^{\frac{3}{2}}}.$$

Substituting in equation (3) Art. 354, and reducing, we have

$$x' = \frac{x^3 (a^2 + b^2)}{a^4}.$$

In a similar manner we obtain, by means of equation (2) above,

$$y' = -\frac{y^3 (a^2 + b^2)}{b^4}.$$

Eliminating x and y from the equation of the hyperbola by means of these results, we derive for the equation of the evolute

$$(ax')^{\frac{2}{3}} - (by')^{\frac{2}{3}} = (a^2 + b^2)^{\frac{2}{3}}.$$

357. By substituting $\frac{ds}{d\phi}$ for ρ in equations (1) and (2) of Art. 354, we obtain the formulas

$$x' - x = -\frac{dy}{d\phi}, \text{ and } y' - y = \frac{dx}{d\phi}, \quad . \quad . \quad . \quad (1)$$

which may frequently be employed with advantage. For example, in the case of the cycloid, we have

$$x = a(\psi - \sin\psi), \qquad y = a(1 - \cos\psi). \quad . \quad . \quad (2)$$

Differentiating,

$$dx = a(1 - \cos\psi)\, d\psi, \quad \text{and} \quad dy = a \sin\psi\, d\psi;$$

whence $$\frac{dy}{dx} = \frac{\sin\psi}{1 - \cos\psi} = \cot\frac{\psi}{2} = \tan\phi,$$

therefore $$\phi = \tfrac{1}{2}\pi - \tfrac{1}{2}\psi, \quad \text{and} \quad d\phi = -\tfrac{1}{2}d\psi.$$

Substituting in formulas (1), we obtain

$$x' - x = 2a\sin\psi, \qquad y' - y = -2a(1 - \cos\psi);$$

and, eliminating x and y by equations (2), we have the equations of the evolute

$$x' = a(\psi + \sin\psi), \qquad y' = -a(1 - \cos\psi).$$

If we transfer the origin to the point $(-a\pi, -2a)$, and denote by x and y the coordinates with reference to the new origin, we shall have

$$x = x' + a\pi \quad \text{and} \quad y = y' + 2a;$$

whence, denoting $\psi + \pi$ by ψ', we obtain the equations

$$x = a(\psi' - \sin\psi'), \qquad y = a(1 - \cos\psi'),$$

which are identical in form with equations (1).

Therefore, *the evolute of the cycloid is an equal cycloid* situated below the axis of x, and having its vertex coincident with a cusp of the given cycloid.

The Length of an Arc of the Evolute.

358. It follows from the mode in which a curve may be generated from its evolute, as explained in Art. 353, that *any continuous arc of the evolute is equal to the difference between the corresponding values of ρ.*

For example, the values of ρ corresponding to the vertex and to the cusp of the evolute determined in the preceding article are respectively zero and $4a$; hence, the length of one half of the evolute is $4a$. It follows therefore, since the evolute is a cycloid equal to the given cycloid, that the length of an entire branch of the cycloid is *eight times the radius of the generating circle.*

359. It is evident that the point of the evolute which corresponds to a maximum or to a minimum value of ρ will be a cusp. Thus, since the radius of curvature of an ellipse is a minimum at each extremity of the major axis, and a maximum at each extremity of the minor axis, the evolute of this curve consists of four branches separated by cusps.

The entire length of a curve of this character can only be found by determining separately the length of each branch.

Involutes and Parallel Curves.

360. The method of generating involutes given in Art. 353 shows that the involutes of a given curve cut its tangents at right angles; in other words, they are trajectories (see Art. 282) of these tangents, and since the constant angle is a right angle they are said to be *orthogonal trajectories.* Hence any two of these involutes are curves having common normals, and it is evident that the portion of the normal intercepted between them is constant. Curves thus related are called *parallel curves.*

To find the general equation of the involute of a given curve requires the aid of the *Integral Calculus;* but when one of the involutes is known, the problem is reduced to that of finding the general equation of the parallels to this curve.

If $PP' = c$ be the constant intercept on the normal, taken in the direction $\phi + 90°$, we have, by projecting PP' upon the axes,

$$\left.\begin{aligned} x' - x &= -c\sin\phi \\ y' - y &= c\cos\phi \end{aligned}\right\}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

in which (x, y) is a point on the known involute, ϕ the inclination of the curve at this point, and (x', y') the corresponding point on the parallel involute.

The Radius of Curvature at a Cusp and at a Point of Inflexion.

361. Since

$$\rho = \frac{ds}{d\phi},$$

ρ will change sign whenever ds or $d\phi$ changes sign, unless these differentials change sign simultaneously.

When ds changes sign the generating point reverses the direction of its motion abruptly, and forms a cusp; for this reason a cusp is sometimes called a *stationary point.*

When $d\phi$ changes sign the tangent at the generating point changes its direction of rotation, and, if ds does not change sign simultaneously, we have a point of inflexion. The tangent at a point of inflexion is, therefore, sometimes called a *stationary tangent.*

At an ordinary or ceratoid cusp, at which the two branches of the curve lie on opposite sides of the common tangent, it is easy to see that the tangent does not change its direction of

rotation as the generating point passes through the cusp; that is, $d\phi$ does not change sign. At a ramphoid cusp, however, both branches of the curve lying on the same side of the tangent, it is evident that the direction of rotation of the tangent line is reversed as the generating point passes through the cusp; in other words, ds and $d\phi$ change sign simultaneously.

362. It follows from the preceding article that ρ changes sign at a ceratoid cusp and at a point of inflexion; hence at these points *the value of* ρ *must be either zero or infinity*, and consequently the evolute must either pass through the point in question or have an infinite branch to which the normal at this point is an asymptote.

At a ramphoid cusp, however, ρ *does not change sign*, and since its value $\dfrac{ds}{d\phi}$ takes an indeterminate form, $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$, it may have a finite value which is of course the same for both branches of the curve. See Ex. XXXIV, 18, 21.

The Radius of Curvature of the Evolute.

363. Denoting by ρ' the radius of curvature of the evolute, we have

$$\rho' = \frac{ds'}{d\phi'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now it follows, from Art. 358, that the rate of s' is identical with that of ρ, hence

$$ds' = d\rho;$$

and, since the tangent to the evolute is the normal to the given curve,

$$\phi' = \phi + \tfrac{1}{2}\pi, \qquad \text{whence} \qquad d\phi' = d\phi.$$

Substituting in (1), we have

$$\rho' = \frac{d\rho}{d\phi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and substituting in (2) the value of $d\phi$ from the equation $\rho = \dfrac{ds}{d\phi}$,

$$\rho' = \rho\frac{d\rho}{ds}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

also, by eliminating ρ from equation (2), we obtain

$$\rho' = \frac{d^2s}{d\phi^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

364. When in the case of a given curve the relation between ρ and ϕ is readily obtained, equation (2) enables us to express ρ' in terms of ϕ', and the relation thus found sometimes serves to determine the evolute.

For example, in the case of the cardioid

$$r = 4a\sin^2\tfrac{1}{2}\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

it was shown in Art. 347 that $\phi = \frac{3}{2}\theta$ and that $\rho = \dfrac{8a}{3}\sin\tfrac{1}{2}\theta$,

hence
$$\rho = \frac{8a}{3}\sin\tfrac{1}{3}\phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Therefore
$$\rho' = \frac{d\rho}{d\phi} = \frac{8a}{9}\cos\tfrac{1}{3}\phi,$$

but since $\phi = \phi' - \frac{1}{2}\pi$, $\cos\frac{1}{3}\phi = \sin(\frac{1}{2}\pi + \frac{1}{3}\phi) = \sin\frac{1}{3}(\phi' + \pi)$,

hence
$$\rho' = \frac{8a}{9}\sin\tfrac{1}{3}(\phi' + \pi). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Comparing equations (2) and (3) it is evident that the evolute is a cardioid whose axis is one-third of that of the given cardioid, and, since the radius of curvature is zero at the cusp of the given cardioid, the vertex of the evolute coincides with this point.

The length of the evolute may be found by the method of Art. 358. For the semi-perimeter of the evolute is equal to the value of ρ at the vertex of the given cardioid, but by equation (2) this value is $\frac{8}{3}a$, or twice the axis of the evolute. Hence *the perimeter of a cardioid is four times its axis.*

Examples XXXV.

1. Find the equation of the evolute of the *ellipse.*

$$(ax')^{\frac{2}{3}} + (by')^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}.$$

2. Show that the entire length of the evolute of the ellipse is

$$4\left(\frac{a^2}{b} - \frac{b^2}{a}\right).$$

See Art. 359.

3. The equation of the *equilateral hyperbola* being

$$xy = m^2,$$

prove that

$$x' + y' = \frac{m}{2}\left(\frac{m}{x} + \frac{x}{m}\right)^3, \quad \text{and} \quad x' - y' = \frac{m}{2}\left(\frac{m}{x} - \frac{x}{m}\right)^3;$$

and thence derive the equation of the evolute.

$$(x' + y')^{\frac{2}{3}} - (x' - y')^{\frac{2}{3}} = (4m)^{\frac{2}{3}}.$$

4. The equation of the *semi-cubical parabola* being

$$x^3 = ay^2,$$

prove that

$$x' = -x - \frac{9x^2}{2a}, \quad \text{and} \quad y' = 4\left(x + \frac{a}{3}\right)\sqrt{\frac{x}{a}};$$

and thence derive the equation of the evolute.

$$729\, ay'^2 = 16[2a + \sqrt{(a^2 - 18ax')}]^2 [\sqrt{(a^2 - 18ax')} - a].$$

5. Given the equation of the *tractrix* (see Art. 303)

$$x = a \log \frac{a + \sqrt{(a^2 - y^2)}}{y} - \sqrt{(a^2 - y^2)};$$

prove that

$$y' = \frac{a^2}{y}, \quad \text{and} \quad x' = a \log \frac{a + \sqrt{(a^2 - y^2)}}{y},$$

and deduce the equation of the evolute.

$$y' = \frac{a}{2}\left(\varepsilon^{\frac{x'}{a}} + \varepsilon^{-\frac{x'}{a}}\right).$$

6. Given the equation of the *catenary*

$$y = \frac{a}{2}\left(\varepsilon^{\frac{x}{a}} + \varepsilon^{-\frac{x}{a}}\right);$$

prove that

$$y' = 2y, \quad \text{and} \quad x' = x - \frac{y}{2}\left(\varepsilon^{\frac{x}{a}} - \varepsilon^{-\frac{x}{a}}\right),$$

and deduce the equation of the evolute.

$$x' = a \log \frac{y' \pm (y'^2 - 4a^2)^{\frac{1}{2}}}{2a} \mp \frac{y'}{4a}(y'^2 - 4a^2)^{\frac{1}{2}}.$$

7. Find the equation of the evolute of the *four-cusped hypocycloid*

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

Expressing $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of x and y, we obtain

$$x' = x + 3(xy^2)^{\frac{1}{3}}, \quad \text{and} \quad y' = y + 3(x^2y)^{\frac{1}{3}},$$

and thence

$$x' + y' = (x^{\frac{1}{3}} + y^{\frac{1}{3}})^3, \quad \text{and} \quad x' - y' = (x^{\frac{1}{3}} - y^{\frac{1}{3}})^3.$$

Whence the required equation is $\qquad (x'+y')^{\frac{2}{3}} + (x'-y')^{\frac{2}{3}} = 2a^{\frac{2}{3}}.$

8. Find the equation of the evolute of the curve

$$y = (2b - a)\cos\psi - (b - a)\cos^3\psi, \; x = (2a - b)\sin\psi - (a - b)\sin^3\psi.$$

Employ the method illustrated in Art. 357.

$$x'^{\frac{2}{3}} + y'^{\frac{2}{3}} = [2(b - a)]^{\frac{2}{3}}.$$

9. Find the equations of the evolute of the curve

$$x = c\sin 2\psi(1 + \cos 2\psi), \qquad y = c\cos 2\psi(1 - \cos 2\psi).$$

$$x' = 2c(-2\sin\psi\cos 3\psi + \sin 2\psi\cos^2\psi);$$
$$y' = 2c(2\cos\psi\cos 3\psi + \cos 2\psi\sin^2\psi).$$

10. Find the cusps and asymptotes of the evolute of the *lemniscata*

$$r^2 = a^2\cos 2\theta.$$

See Art. 359 *and Art.* 362.

11. Prove, by the method of Art. 364, that the evolute of the *logarithmic spiral*,

$$r = a\varepsilon^{n\theta},$$

is a similar logarithmic spiral, and show that the constant ratio between the corresponding radii of curvature is equal to n.

12. Prove that the evolute of an *epicycloid* having a cusp on the axis of x, the radii of the fixed and rolling circles being a and b respectively, is a similar epicycloid having its vertex on the axis of x, and whose radii are

$$a' = \frac{a^2}{a + 2b} \quad \text{and} \quad b' = \frac{ab}{a + 2b}.$$

13. Find the radius of curvature, and the equation of the evolute of the *cissoid*

$$y^2 = \frac{x^3}{2a - x}.$$

$$\rho = \frac{a(8a - 3x)^{\frac{3}{2}} x^{\frac{1}{2}}}{3(2a - x)^2}, \quad \text{and} \quad 4096a^3x' + 1152a^2y'^2 + 27y'^4 = 0.$$

14. Show, by means of the result obtained in Example 12, that one of the involutes of the epicycloid, the radii of whose fixed and rolling circles are a and b, is the similar epicycloid whose radii are $a + 2b$ and $\dfrac{b(a + 2b)}{a}$; and thence derive the general equation of the involutes. *See Art.* 360.

$$x = \frac{(a + 2b)(a + b)}{a} \cos\psi + \frac{b(a + 2b)}{a} \cos\frac{a + b}{b}\psi - c\cos\frac{a + 2b}{2b}\psi:$$

$$y = \frac{(a + 2b)(a + b)}{a} \sin\psi + \frac{b(a + 2b)}{a} \sin\frac{a + b}{b}\psi - c\sin\frac{a + 2b}{2b}\psi.$$

15. Find the general equations of the involutes of the cycloid. *See Art.* 360.

$$x' = a(\pi + \psi - \sin\psi) - c\cos\tfrac{1}{2}\psi:$$
$$y' = a(3 - \cos\psi) + c\sin\tfrac{1}{2}\psi.$$

XXXVI.

Envelopes.

365. The curves determined by an equation involving x and y together with constants to which arbitrary values may be assigned are said to constitute a *system* of curves. The arbitrary constants are called *parameters*. When but one of

the parameters is regarded as variable, denoting it by α, the general equation of the system of curves may be expressed thus :

$$f(x, y, \alpha) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

When the curves of a system mutually intersect (the intersections not being fixed points), there usually exists a curve which touches each curve of the system obtained by causing the value of α to vary.

For example, the ellipses whose axes are fixed in position, and whose semi-axes have a constant sum, constitute such a system; and, if we regard the ellipse as varying continuously from the position in which one semi-axis is zero to that in which the other is zero, it is evident that the boundary of that portion of the plane which is swept over by the perimeter of the varying ellipse is a curve to which the ellipse is tangent in all its positions. A curve having this relation to a given system of curves is called the *envelope of the system.*

Every point on an envelope may be regarded as the limiting position of the point of intersection of two members of the given system of curves, when the difference between the corresponding values of α is indefinitely diminished. For this reason, the envelope is sometimes called *the locus of the ultimate intersections* of the curves of the given system.

366. If we differentiate equation (1) of the preceding article (regarding α as a variable as well as x and y) the resulting equation will be of the general form

$$f'_x(x, y, \alpha)\, dx + f'_y(x, y, \alpha)\, dy + f'_\alpha(x, y, \alpha)\, d\alpha = 0. \quad . \quad (2)$$

In this equation each term may be separately obtained by differentiating the given equation on the supposition that the quantity indicated by the subscript is alone variable. See Art. 64.

From equation (2) we derive

$$\frac{dy}{dx} = -\frac{f'_x(x, y, \alpha)}{f'_y(x, y, \alpha)} - \frac{f'_\alpha(x, y, \alpha)}{f'_y(x, y, \alpha)}\frac{d\alpha}{dx}. \quad . \quad . \quad . \qquad (3)$$

In Fig. 77 let PC be the curve corresponding to a particular value of α, and let P be the point (x, y); then the expression for $\frac{dy}{dx}$ given in equation (3) determines the direction in which the point P is actually moving when x, y, and α vary simultaneously. This direction depends therefore in part upon the arbitrary value given to the ratio $\frac{d\alpha}{dx}$.

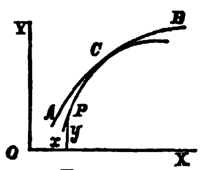


FIG. 77.

367. Now if α were constant $d\alpha$ would vanish, and equation (3) would become

$$\frac{dy}{dx} = -\frac{f'_x(x, y, \alpha)}{f'_y(x, y, \alpha)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

This expression for $\frac{dy}{dx}$ determines the direction in which P moves when PC is a fixed curve.

Let AB be an arc of the envelope, and let C be its point of contact with PC. Now, if P be placed at the point C, it is obvious that it can move only in the direction of the common tangent at C, *whether α be fixed or variable.* It follows therefore that, at every point at which a curve belonging to the system touches the envelope, the expressions for $\frac{dy}{dx}$ given in equations (3) and (4) must be identical in value.

Assuming that $f'_x(x, y, \alpha)$ and $f'_y(x, y, \alpha)$ do not become infinite for any finite values of x and y, the above condition requires that

$$f'_\alpha(x, y, \alpha) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

Hence the coordinates of every point of the envelope must satisfy simultaneously equations (5) and (1); the equation of the envelope is therefore obtained by eliminating α between these two equations.

368. Let it be required to find the *envelope of the circles having for diameters the double ordinates of the parabola*

$$y^2 = 4ax.$$

If we denote by α the abscissa of the centre of the variable circle, its radius will be the ordinate of the point on the parabola of which α is the abscissa, the equation of the circle will therefore be

$$y^2 + (x - \alpha)^2 - 4a\alpha = 0. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Differentiating with reference to the variable parameter α, we have

$$-2(x - \alpha) - 4a = 0,$$

or

$$\alpha = 2a + x; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

substituting in (1), and reducing, we obtain

$$y^2 = 4a(a + x). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The envelope is, therefore, a parabola equal to the given parabola and having its focus at the vertex of the given parabola.

369. In the above example every point of the envelope is a point of contact with a member of the given system of circles; for the value of α corresponding to any given point of the envelope is always real, since it is determined by means of equation (2). This equation serves also to determine the abscissa of the points corresponding to a given value of α; but, when α is less than a, the ordinates of these points, determined by equation (3), are found to be imaginary. Hence a portion of the system of curves given by equation (1) does

not admit of an envelope; in fact, the circles belonging to this part of the system do not intersect.

On the other hand, examples are sometimes met with in which the value of α corresponding to certain real points on the envelope is found to be imaginary.. When this is the case, the envelope contains an arc or arcs which do not properly belong to it as geometrically defined.

Fixed Points of Intersection.

370. When the equation of a system of curves is in a rational integral form with respect to x, y, and α, the *order* of the system is indicated by the degree of the equation with respect to x and y, and the *index* of the system by its degree with reference to α.

The equation of a system whose index is unity may be written in the form

$$F_1(x, y) + \alpha F_2(x, y) = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Every point which satisfies the simultaneous equations

$$F_1(x, y) = 0 \qquad \text{and} \qquad F_2(x, y) = 0, \quad . \quad . \quad . \quad (2)$$

evidently satisfies equation (1) for all values of α: hence all the curves of the given system pass through the fixed points in which the curves (2) intersect.

Moreover, two curves of the given system cannot intersect in any other points; for, at a point common to any two curves of the system, we have

$$F_1(x, y) + \alpha_1 F_2(x, y) = 0, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and

$$F_1(x, y) + \alpha_2 F_2(x, y) = 0. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

By subtracting we obtain

$$(\alpha_1 - \alpha_2) F_2(x, y) = 0,$$

$\therefore \qquad F_2(x, y) = 0$ and consequently $F_1(x, y) = 0$;

hence every intersection of the curves (3) and (4) coincides with one of the fixed points determined by equations (2).

A general equation of the form (1), therefore, represents a system of curves which intersect only in fixed points and consequently does not admit of an envelope. Such a system is called a *pencil* of curves.

371. If the equation of a system of curves can be put in the form

$$F_1(x, y) + f(x, y, \alpha)\cdot F_2(x, y) = 0, \quad . \quad . \quad . \quad . \quad (1)$$

all the curves will pass through the intersections of

$$F_1(x, y) = 0 \quad \text{and} \quad F_2(x, y) = 0; \quad . \quad . \quad . \quad . \quad (2)$$

but in this case the curves of the system may also intersect in other points, and hence *the system may have an envelope.*

By differentiating equation (1) with reference to α we derive

$$f'_\alpha(x, y, \alpha)\cdot F_2(x, y) = 0, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which is satisfied by $\quad F_2(x, y) = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$

and also by $\quad f'_\alpha(x, y, \alpha) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$

Equation (4) indicates the existence of the fixed points of intersection, and equation (5) determines the envelope.

372. For example, let it be required to find *the envelope of the series of parabolas obtained by varying the angle of elevation of a projectile having a given initial velocity.*

The equation of the path of a projectile, when no allowance is made for the resistance of the air, is

$$y = x \tan \alpha - \frac{x^2}{4h \cos^2\alpha},$$

the origin being at the point of projection, and α denoting the angle of elevation above the horizontal plane. Putting α in place of tan α, we obtain

$$4h(y - \alpha x) + (1 + \alpha^2)x^2 = 0,$$

or $$4hy + x^2 + x(x\alpha^2 - 4h\alpha) = 0. \quad . \quad . \quad . \quad . \quad (1)$$

Taking the derivative, we have

$$2x^2(x\alpha - 2h) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The solution $x = 0$ indicates the existence of fixed points of intersection in the system of curves. The equation $F_1(x, y) = 0$ is, in this case,

$$4hy + x^2 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The fixed points of intersection are therefore the points common to the line $x = 0$, and the curve given by equation (3). These points are the origin and the point at infinity on the parabola $4hy + x^2 = 0$.

The other solution of equation (2), viz.,

$$x\alpha - 2h = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

determines the envelope. Substituting in equation (1) the value of α obtained from equation (4), we derive

$$x^2 = 4h(h - y) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

for the equation of the envelope, which is, therefore, a parabola with its axis vertical and its focus at the origin.

Equations in an Irrational Form.

373. It is assumed in Art. 367 that f'_x and f'_y cannot be infinite for finite values of x and y. Cases in which these functions become infinite cannot arise when the given equation is

rational in form; but when an irrational form of the given equation is employed, a relation between x and y can sometimes be found, which gives

$$f'_x(x, y, \alpha) = \infty \quad \text{and} \quad f'_y(x, y, \alpha) = \infty,$$

and does not at the same time give an infinite value to $f'_\alpha(x, y, \alpha)$.

The value of $\dfrac{dy}{dx}$ determined by equation (3), Art. 366, will, in such cases, reduce to

$$-\frac{f'_x(x, y, \alpha)}{f'_y(x, y, \alpha)},$$

which is identical with the value given by equation (4), Art. 367: hence the points determined by a relation of this kind belong to the envelope.

374. For example, if the given equation is

$$y - 2ax \pm \sqrt{(y^2 - 4ax)} = 0, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

we have

$$f'_y = 1 \pm \frac{y}{\sqrt{(y^2 - 4ax)}} \quad \text{and} \quad f'_x = -2a \mp \frac{2a}{\sqrt{(y^2 - 4ax)}}.$$

These expressions are both infinite when

$$y^2 - 4ax = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

while f'_α is finite. Hence equation (2) (since it does not contain α) is the equation of an envelope.

375. By putting $f'_\alpha = 0$ in this example, we obtain

$$x = 0;$$

this line does not, however, properly belong to the envelope, as will appear from the rationalized form of equation (1), viz.,

$$\alpha^2 x^2 - \alpha xy + ax = 0,$$

the locus of which is composed of the two straight lines

$$x = 0 \quad \text{and} \quad y - \alpha x - \frac{a}{\alpha} = 0.$$

Since the first of these equations is independent of α it constitutes a *fixed branch* of the given system.

Whenever the variable system of lines contains a fixed branch, the motion of a point situated on this branch will be independent of α; hence such points satisfy the condition employed in Art. 367 to determine an envelope.

The envelope of the variable line

$$y - \alpha x - \frac{a}{\alpha} = 0$$

(obtained from this form of the equation by the condition $f'_\alpha = 0$) is of course the parabola whose equation is found above.

Two Variable Parameters.

376. When the equation of the given curve contains two variable parameters connected by an equation, only one of these parameters can be regarded as arbitrary, since, by means of the equation connecting them, one of the parameters can be eliminated. Instead, however, of eliminating one of the parameters at once, it is often better to proceed as in the following example.

Required, to find *the envelope of a varying circle whose centre moves on a given circle, and whose circumference is always tangent to a fixed diameter of the given circle.*

Denoting the coordinates of the centre of the varying circle

by α and β, the centre of the fixed circle being the origin, and the fixed diameter being the axis of x, the equation of the variable circle is $(x - \alpha)^2 + (y - \beta)^2 = \beta^2$,

or $$(x - \alpha)^2 + y^2 - 2\beta y = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Denoting the radius of the fixed circle by a we have also

$$\alpha^2 + \beta^2 = a^2; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

differentiating (1), we have

$$(x - \alpha)\, d\alpha + y d\beta = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and from (2) $$\alpha d\alpha + \beta d\beta = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

We have now four equations from which to eliminate α, β, and the ratio $\frac{d\alpha}{d\beta}$. Transposing, and dividing (3) by (4) to eliminate $\frac{d\alpha}{d\beta}$, we have

$$\frac{x - \alpha}{\alpha} = \frac{y}{\beta}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

substituting in equation (1) the value of $x - \alpha$,

$$\frac{\alpha^2 y^2}{\beta^2} + y^2 = 2\beta y,$$

or, by equation (2) $$\frac{a^2 y^2}{\beta^2} = 2\beta y.$$

Whence $$y = 0, \quad \text{and} \quad \tfrac{1}{2} a^2 y = \beta^3. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$y = 0$ is the equation of the diameter of the fixed circle, which evidently constitutes a part of the envelope.

From equation (6), we have

$$\beta = \frac{a^{\frac{2}{3}} y^{\frac{1}{3}}}{2^{\frac{1}{3}}},$$

and, from equation (5),

$$\alpha = \frac{x\beta}{y + \beta} = \frac{a^{\frac{2}{3}}x}{a^{\frac{2}{3}} + 2^{\frac{1}{3}}y^{\frac{2}{3}}}.$$

Substituting these values of α and β in equation (2),

$$\frac{x^2}{(a^{\frac{2}{3}} + 2^{\frac{1}{3}}y^{\frac{2}{3}})^2} + \frac{y^{\frac{2}{3}}}{2^{\frac{2}{3}}} = a^{\frac{2}{3}};$$

whence, clearing of fractions and reducing,

$$4^{\frac{1}{3}}(x^2 + y^2 - a^2) = 3a^{\frac{4}{3}}y^{\frac{2}{3}},$$

or

$$4(x^2 + y^2 - a^2)^3 = 27a^4y^2.$$

This is the equation of the *two-cusped epicycloid.* See Art. 294.

The Application of Undetermined Multipliers to Envelopes.

377. In some cases the elimination of the parameters is facilitated by the application of the method of *undetermined multipliers* as illustrated in the following example.

A variable circle moves with its centre on the circumference of a fixed circle, while its radius has a given ratio to the chord joining its centre to a fixed point of the circumference.

Taking the fixed point as an origin, and a diameter of the circle as the axis of x, the equation of the fixed circle is

$$x^2 + y^2 - 2ax = 0.$$

Let α and β denote the coordinates of the centre of the variable circle, and n the given ratio of its radius to the chord. The length of the chord being $\sqrt{(\alpha^2 + \beta^2)}$, we have, for the equation of the variable circle,

$$(x - \alpha)^2 + (y - \beta)^2 = n^2(\alpha^2 + \beta^2),$$

and, for the relation between the parameters,

$$\alpha^2 + \beta^2 - 2a\alpha = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

By substituting the value of $\alpha^2 + \beta^2$ from equation (1), the equation of the variable circle becomes

$$x^2 + y^2 - 2\alpha x - 2\beta y - 2(n^2 - 1)a\alpha = 0. \quad . \quad . \quad (2)$$

Hence the differential equations are

$$(\alpha - a)\,d\alpha + \beta d\beta = 0,$$

and
$$[x + (n^2 - 1)a]\,d\alpha + y d\beta = 0.$$

Multiplying the second by λ, and subtracting from the first,

$$[\alpha - a - \lambda\{x + (n^2 - 1)a\}]\,d\alpha + (\beta - \lambda y)\,d\beta = 0.$$

Since λ is arbitrary, we are at liberty to assume the coefficient of $d\alpha$ to be zero, and, as a consequence of this assumption, we have the coefficient of $d\beta$ likewise equal to zero; whence

$$\alpha - a - \lambda[x + (n^2 - 1)a] = 0, \quad . \quad . \quad . \quad . \quad (3)$$

and
$$\beta - \lambda y = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

We have now to eliminate α, β, and λ between equations (1), (2), (3), and (4). To effect this elimination, we multiply (3) by α, (4) by β, and add, thus obtaining

$$\alpha^2 + \beta^2 - a\alpha - \lambda(\alpha x + \beta y + (n^2 - 1)a\alpha) = 0.$$

Simplifying this equation by means of (1) and (2), we obtain

$$a\alpha - \tfrac{1}{2}\lambda(x^2 + y^2) = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

eliminating α between (3) and (5),

$$\lambda = \frac{2a^2}{x^2 + y^2 - 2ax - 2(n^2 - 1)a^2};$$

whence, from equation (5),

$$\alpha = \frac{a(x^2+y^2)}{x^2+y^2-2ax-2(n^2-1)a^2},$$

and, from equation (4),

$$\beta = \frac{2a^2y}{x^2+y^2-2ax-2(n^2-1)a^2}.$$

By substituting these values of α and β in equation (2), we derive

$$x^2+y^2-2\,\frac{ax(x^2+y^2)+(n^2-1)a^2(x^2+y^2)+2a^2y^2}{x^2+y^2-2ax-2(n^2-1)a^2}=0,$$

which reduces to

$$(x^2+y^2)^2-4ax(x^2+y^2)-4(n^2-1)a^2(x^2+y^2)-4a^2y^2=0,$$

or $$(x^2+y^2)^2-(4ax+4n^2a^2)(x^2+y^2)+4a^2x^2=0.$$

The envelope is therefore a limaçon. See Art. 269.

Examples XXXVI.

✓ 1. Find the envelope of the system of parabolas represented by the equation

$$y^2 = \frac{\alpha^2}{c}(x-\alpha),$$

in which α is an arbitrary parameter and c a fixed constant.

$$y^2 = \frac{4}{27c}x^3.$$

✓ 2. Find the envelope of the circles described on the double ordinates of an ellipse as diameters.

$$\frac{x^2}{a^2+b^2}+\frac{y^2}{b^2}=1.$$

3. Find the envelope of the ellipses, the product of whose semi-axes is equal to the constant c^2.

The conjugate hyperbolas, $2xy = \pm c^2$.

4. Find the envelope of a perpendicular to the normal to the parabola, $y^2 = 4ax$, drawn through the intersection of the normal with the axis.

$$y^2 = 4a\,(2a - x).$$

5. Find the envelope of the ellipses whose axes are fixed in position, and whose semi-axes have a constant sum c.

The four-cusped hypocycloid, $x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}$.

6. A circle moves with its centre on a parabola whose equation is $y^2 = 4ax$, and passes through the vertex of the parabola; find the envelope.

The cissoid, $y^2\,(x + 2a) + x^3 = 0$.

7. A straight line cuts the coordinate axes in such a manner that the product of the intercepts is constant and equal to c^2; find the envelope.

$$xy = \tfrac{1}{4}c^2.$$

8. A perpendicular to the tangent to a parabola is drawn at the point where the tangent cuts the fixed line $x = c$; find the equation of the envelope of this perpendicular.

The parabola, $y^2 = -4c\,(x - a - c)$.

9. Prove that the envelope of the system of curves

$$y - \alpha x \pm \sqrt{[f(x, y)]} = 0 \qquad \text{is} \qquad f(x, y) = 0.$$

10. Show that the hyperbolas given by the equation

$$\frac{\alpha}{x} + \frac{\beta}{y} = 1, \qquad \text{when} \qquad \alpha + \beta = c,$$

form a pencil, and therefore do not admit of an envelope.

11. The centre of an hyperbola passing through the origin, and having asymptotes parallel to the axes moves upon the circle $x^2 + y^2 = a^2$; find the envelope.

$$x^2y^2 = a^2(x^2 + y^2).$$

12. A straight line of fixed length a moves with its extremities in two rectangular axes; find the envelope.

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

13. Show that the envelope of

$$A\alpha^2 + B\alpha + C = 0,$$

where A, B, and C are functions of x and y, is

$$4AC - B^2 = 0.$$

14. Find the curve to which the lines given by the equation

$$y = mx \pm \sqrt{(am^2 + bm + c)}$$

are tangent.

$$4(ay^2 + bxy + cx^2) = 4ac - b^2.$$

15. Show that the envelope of

$$A\alpha^3 + B\alpha^2 + C\alpha + D = 0,$$

A, B, C, and D being functions of x and y, is

$$18ABCD - 27A^2D^2 + B^2C^2 - 4(AC^3 + B^3D) = 0.$$

16. The centre of a circle which passes through the origin moves on the equilateral hyperbola

$$x^2 - y^2 = a^2;$$

find the envelope.

The lemniscata, $(x^2 + y^2)^2 = 4a^2(x^2 - y^2)$.

17. Find the envelope of the parabolas which touch the coordinate axes at the distances α and β from the origin, when $\alpha\beta = c^2$, the equation of the parabola being

$$\frac{x^{\frac{1}{2}}}{\alpha^{\frac{1}{2}}} + \frac{y^{\frac{1}{2}}}{\beta^{\frac{1}{2}}} = 1. \qquad 16xy = c^2.$$

18. The intercepts, α and β, of a straight line on the coordinate axes are connected by the linear relation

$$n\alpha + \beta = c;$$

find the envelope.

The parabola, $(y - nx)^2 - 2ncx - 2cy + c^2 = 0$.

19. Find the envelope of the system of curves,

$$\frac{x^n}{\alpha^n} + \frac{y^n}{\beta^n} = 1,$$

when $$\alpha^m + \beta^m = c^m.$$

$$x^{\frac{mn}{m+n}} + y^{\frac{mn}{m+n}} = c^{\frac{mn}{m+n}}.$$

20. By means of the result obtained in example 19, show that the envelope of a straight line, the sum of whose intercepts is constant, is a parabola touching the coordinate axes.

21. From a point in the ellipse perpendiculars are drawn to the axes; find the envelope of the line joining the feet of these perpendiculars.

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1.$$

XXXVII.

Envelopes of Systems of Straight Lines.

378. In the case of envelopes of systems of straight lines, expressions for the coordinates in terms of the arbitrary parameter can always be found, and, although the elimination of this parameter may be impracticable, the curve can always be traced; as the following example will serve to show.

A line AB *of fixed length* a *slides between rectangular axes: it is required to determine the envelope of a perpendicular to this line passing through the extremity* A *that slides on the axis of* x.

Denoting by α the angle between AB and the axis of x, we have for the intercept OA the expression $a \cos \alpha$, and for the equation of the perpendicular

$$y = \cot \alpha (x - a \cos \alpha),$$

or

$$y \tan \alpha - x + a \cos \alpha = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Taking the derivative with reference to α, we have

$$y \sec^2 \alpha - a \sin \alpha = 0,$$

or

$$y = a \sin \alpha \cos^2 \alpha; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

whence, substituting in (1), we derive

$$x = a \sin^2 \alpha \cos \alpha + a \cos \alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Equations (2) and (3) give x and y in terms of α, and hence determine the curve. It is evident that the curve is symmetrical to both axes, since $\pm\, \alpha$ and $\pi \pm \alpha$ determine values of x and y differing only in sign.

By differentiation, we obtain

$$\frac{dy}{d\alpha} = a \cos \alpha (\cos^2 \alpha - 2 \sin^2 \alpha), \quad . \quad . \quad . \quad . \quad (4)$$

and

$$\frac{dx}{d\alpha} = a \sin \alpha (\cos^2 \alpha - 2 \sin^2 \alpha). \quad . \quad . \quad . \quad . \quad (5)$$

Corresponding to $\alpha = 0$, we have the point $(a, 0)$ at which the curve cuts the axis of x at right angles. As α increases from zero, equations (4) and (5) show that x and y increase until

$$\cos^2 \alpha - 2 \sin^2 \alpha = 0 \qquad \text{or} \qquad \tan \alpha = \tfrac{1}{2} \sqrt{2},$$

and that they subsequently decrease simultaneously; the point

corresponding to this value of α is therefore a cusp. The arc generated as α passes from this value to $90°$, touches the axis of x at the origin.

The Evolute regarded as an Envelope.

379. Since the evolute of a given curve is the curve to which all the normals to the given curve are tangent, it is evidently the envelope of these normals.

The equation of the normal at the point (x, y) of a given curve may be written in the form

$$x' - x + (y' - y)\frac{dy}{dx} = 0, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

(x', y') being any point of the normal.

In this equation y and $\frac{dy}{dx}$ are functions of x determined by the equation of the given curve, and x is to be regarded as the arbitrary parameter. Hence, differentiating with reference to x, we have

$$-1 - \left(\frac{dy}{dx}\right)^2 + (y' - y)\frac{d^2y}{dx^2} = 0. \quad . \quad . \quad . \quad . \quad (2)$$

Equations (1) and (2) are equivalent to the equations given in Art. 354. Hence it is only when the equation of the normal is expressed in terms of some other parameter that this method differs essentially from that previously given.

380. To illustrate, let us determine *the evolute of the ellipse*, employing the equation of the normal in terms of the eccentric angle.

The equations of the ellipse are

$$x = a\cos\phi, \qquad \text{and} \qquad y = b\sin\phi;$$

whence $\quad dx = -a\sin\phi\, d\phi, \quad$ and $\quad dy = b\cos\phi\, d\phi.$

Substitution in the equation of the normal,

$$(x' - x)\,dx + (y' - y)\,dy = 0,$$

gives $$ax' \sin\psi - by' \cos\psi - (a^2 - b^2)\sin\psi\cos\psi = 0.$$

Differentiating, we have

$$ax' \cos\psi + by' \sin\psi - (a^2 - b^2)(\cos^2\psi - \sin^2\psi) = 0;$$

eliminating y' and x' successively, and dropping the accents,

$$ax = (a^2 - b^2)\cos^3\psi \qquad \text{and} \qquad by = -(a^2 - b^2)\sin^3\psi;$$

whence $$(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}.$$

Negative Pedals.

381. In Fig. 78 P, the foot of the perpendicular from the origin upon the tangent PP' to a curve, describes the pedal of this curve; hence the curve to which PP' is tangent is the *negative pedal* (Art. 308) of the curve described by P. Hence *the negative pedal of a given curve is the envelope of the perpendicular drawn through the extremity of the radius vector from the pedal origin.*

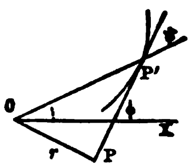


FIG. 78.

If θ, the vectorial angle of the point P on the given curve, be taken as the arbitrary parameter the equation of the perpendicular will be

$$x\cos\theta + y\sin\theta = r, \quad \ldots \ldots \quad (1)$$

whence $$-x\sin\theta + y\cos\theta = \frac{dr}{d\theta}; \quad \ldots \ldots \quad (2)$$

and, eliminating y and x successively, we obtain

$$\left.\begin{aligned} x &= r\cos\theta - \frac{dr}{d\theta}\sin\theta \\ y &= r\sin\theta + \frac{dr}{d\theta}\cos\theta \end{aligned}\right\}^* \quad . \quad . \quad . \quad . \quad . \quad (3)$$

the rectangular coordinates of the negative pedal.

382. For example, let it be required *to determine the negative pedal of the strophoid, the node being the pedal origin.*

The polar equation of the strophoid referred to its node is [see equation (4), Art. 267]

$$r = \frac{a\cos 2\theta}{\cos\theta} = a(\cos\theta - \sin\theta\tan\theta);$$

whence $$\frac{dr}{d\theta} = -a\sin\theta(2 + \sec^2\theta).$$

Substituting in equations (3) of the preceding article and reducing, we have

$$x = a\sec^2\theta, \quad \text{and} \quad y = -2a\tan\theta;$$

whence, eliminating θ,

$$y^2 = 4a(x - a).$$

Hence the negative pedal is a parabola whose vertex is situated at the point $(a, 0)$, the vertex of the strophoid.

* Since it is shown in Art. 350 that the values of ψ are identical at corresponding points of a curve and its pedal, the value of PP' is $r\cot\psi$; hence these expressions for x and y could have been derived by projecting the broken line OPP' on the coordinate axes.

Reciprocal Curves.

383. The reciprocal of a given curve is defined in Art. 309 as the locus of the pole of the tangent to the given curve; that is, PR, Fig. 79, being tangent to the given curve, the reciprocal is the locus of P' when

$$OR \cdot OP' = k^2.$$

The locus of R is the *pedal* (see Art. 307) of the given curve, while the locus of P' is the *inverse* (see Art. 306) of that of R. Now it was shown in Art. 350 that the values of ψ (the inclination of the curve to its radius vector) are identical at P and R, these being corresponding points of a curve and its pedal. Again, since the radius vector of the inverse of a given curve is k^2u, it is obvious, on comparing equation (3) of Art. 318 with equation (6) of Art. 319, that the values of ψ at R and P' are supplementary; hence the values of ψ at P and P' are supplementary.

384. We shall now show that *the given curve is the reciprocal of its own reciprocal.* In other words, it is to be proved that P is the pole of the tangent to the reciprocal curve at P'.

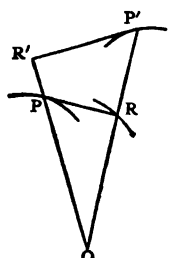


FIG. 79.

Let this tangent meet OP in R'. The angles OPR and $OP'R'$ are equal by the preceding article; hence the angle at R' is a right angle, and the similar triangles give

$$OP \cdot OR' = OR \cdot OP' = k^2$$

Therefore P is the pole of $P'R$, and the locus of P is the reciprocal of the locus of P'.

It follows that *the reciprocal may be defined as the envelope of the polar of a point on the given curve, or as the negative pedal of the inverse of the given curve.*

385. If r' and θ' denote the polar coordinates of P', we have

$$x' = r'\cos\theta', \qquad y' = r'\sin\theta', \qquad OR = \frac{k^2}{r'};$$

and the rectangular coordinates of R are

$$\frac{k^2\cos\theta'}{r'} \quad \text{and} \quad \frac{k^2\sin\theta'}{r'}.$$

The equation of PR, since it passes through the point R and is inclined to the axis of x at the angle $\theta + \frac{1}{2}\pi$, is

$$y - \frac{k^2\sin\theta'}{r'} = -\cot\theta'\left(x - \frac{k^2\cos\theta'}{r'}\right),$$

or $\quad r'y\sin\theta' - k^2\sin^2\theta' + r'x\cos\theta' - k^2\cos^2\theta' = 0;$

that is $\quad xx' + yy' = k^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$

which is therefore the equation of the polar of P'.

The reciprocal may now be regarded either as the locus of P', when the line (1) is tangent to the given curve; or as the envelope of the line (1), when P' is a point of the given curve. When the latter view is adopted x' and y' must be regarded as two parameters connected by the equation of the given curve, and the envelope is found as in Art. 376. See Examples 14 and 15, below.

386. *The reciprocal of a conic with reference to any point is a conic.*

Adopting the former of the two methods suggested in the last paragraph of the preceding article, we take the given point as an origin and denote by P_1 a point of the given conic. The equation of the given conic may be assumed in the homogeneous form

$$Ax_1^2 + 2Bx_1y_1 + Cy_1^2 + 2Dax_1 + 2Eay_1 + Fa^2 = 0.$$

The equation of the tangent at P_1 is, by equation (4), Art. 313,

$$x(Ax_1 + By_1 + Da) + y(Bx_1 + Cy_1 + Ea) + a(Dx_1 + Ey_1 + Fa) = 0. \quad (1)$$

Denoting the pole of this tangent by P_2, and putting $-ab$ in place of k^2 in formula (1) of Art. 385, its equation may also be written in the form

$$xx_2 + yy_2 + ab = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now, since (1) and (2) are equations of the same line, the coefficients of x, y, and a in these equations must have a common ratio, and, denoting this ratio by l, we may therefore write

$$\left.\begin{aligned} Ax_1 + By_1 + Da &= lx_2, \\ Bx_1 + Cy_1 + Ea &= ly_2, \\ Dx_1 + Ey_1 + Fa &= lb. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

These equations when solved for x_1, y_1, and a will obviously give expressions for these quantities of the first degree with respect to x_2, y_2, and b.

Now, since $x_1\,y_1$ is on the line (2), we have

$$x_1x_2 + y_1y_2 + ab = 0,$$

and, if in this equation we substitute the linear expressions for x_1, y_1, and a, the result will obviously be of the second degree with respect to x_2, y_2, and b; hence *the reciprocal curve is a conic.**

* If we denote by Δ the symmetrical determinant of equations (3), and by α, β, etc., the minors corresponding to A, B, etc., the solution of these simultaneous equations is

$$\begin{aligned} \Delta x_1 &= l\,(\alpha x_2 + \beta y_2 + \delta b), \\ \Delta y_1 &= l\,(\beta x_2 + \gamma y_2 + \varepsilon b), \\ \Delta a &= l\,(\delta x_2 + \varepsilon y_2 + \varphi b); \end{aligned}$$

and the substitution gives

$$\alpha x_2^2 + 2\beta x_2 y_2 + \gamma y_2^2 + 2\delta b x_2 + 2\varepsilon b y_2 + \varphi b^2 = 0.$$

Caustics.

387. When a system of rays of light in a plane is reflected or refracted at a given curve, the reflected or refracted rays will in general form, not a pencil of rays passing through a common point, but a system having an envelope. This envelope is called a *caustic* of the given curve; the point from which the rays emanate is called the *radiant point.*

When the caustic has a cusp, a large number of the rays pass nearly through this point; hence the cusp of a caustic is called the *focus* (burning point) corresponding to the given position of the radiant point.

388. *To find the caustic by reflection of a circle when the radiant point is on the circumference.*

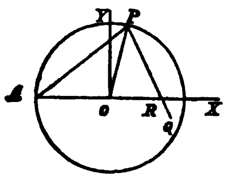


FIG. 80.

Let AP represent an incident and PQ a reflected ray. Denoting the angle PAO by θ, we have, evidently

$$APQ = 2\theta, \qquad \text{and} \qquad PRX = 3\theta.$$

The coordinates of P are

$$a \cos 2\theta \qquad \text{and} \qquad a \sin 2\theta;$$

hence the equation of PQ is

$$y - a \sin 2\theta = \tan 3\theta\,(x - a \cos 2\theta),$$

or
$$\begin{aligned} y \cos 3\theta - x \sin 3\theta &= -a\,(\sin 3\theta \cos 2\theta - \cos 3\theta \sin 2\theta) \\ &= -a \sin \theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1) \end{aligned}$$

Taking derivatives, we deduce

$$y \sin 3\theta + x \cos 3\theta = \tfrac{1}{3}a \cos \theta; \quad . \quad . \quad . \quad . \quad (2)$$

eliminating y from (1) and (2), we express x in terms of θ; thus

$$x = \tfrac{1}{3}a \cos 3\theta \cos \theta + a \sin 3\theta \sin \theta.$$

This equation may be simplified in the following manner:

$$x = (\tfrac{2}{3}a - \tfrac{1}{3}a)\cos 3\theta \cos\theta + (\tfrac{2}{3}a + \tfrac{1}{3}a)\sin 3\theta \sin\theta;$$

whence
$$x = \tfrac{2}{3}a \cos 2\theta - \tfrac{1}{3}a \cos 4\theta. \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

In like manner, we find

$$y = \tfrac{2}{3}a \sin 2\theta - \tfrac{1}{3}a \sin 4\theta. \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

Equations (3) and (4) show that the caustic is an *epicycloid;* the radius of the rolling circle, and also that of the fixed circle, being $\frac{1}{3}a$: but it is shown in Art. 296 that in this case the epicycloid becomes a *cardioid.*

Examples XXXVII.

1. From any point C on the circumference of a circle whose radius is a an ordinate to the fixed diameter AB is drawn, and through the foot of the ordinate a perpendicular to the chord AC is drawn; find the equations of the envelope of this perpendicular, and trace the curve.

Denoting by α the angle BAC, and taking the origin at A, we derive

$$x = 2a\cos^2\alpha\,(3 - 2\cos^2\alpha):$$
$$y = -\,4a\sin\alpha\cos^3\alpha.$$

The curve is symmetrical to the axis of x. $\alpha = 0$ gives the point $(2a, 0)$, $\alpha = 30°$ gives a cusp, and $\alpha = 90°$ a cusp at the origin.

2. From any point B on the circumference of a circle whose radius is a an ordinate is drawn to the fixed diameter AC, and the foot of the ordinate is joined to the middle point of the chord AB; find the envelope of the joining line.

Denoting the angle BAC by θ, and taking the origin at A, we derive

$$x = 2a\cos^2\theta\cos 2\theta:$$
$$y = 2a\sin^2\theta\sin 2\theta.$$

This curve, like the curve determined in the preceding example, is the three-cusped hypocycloid, the origin being in this case a vertex.

3. The points A and B move uniformly on the circumference of the circle whose radius is a, the ratio of the rates of these points being $m : n$; find the envelope of the chord AB.

Denoting by $m\psi$ and $n\psi$ the inclinations of the radii OA and OB, respectively, to the axis of x, the equation of the chord is

$$x \cos \tfrac{1}{2}(m+n)\psi + y \sin \tfrac{1}{2}(m+n)\psi = a \cos \tfrac{1}{2}(m-n)\psi;$$

whence we derive the equations of the envelope; viz.,

$$x = \frac{am}{m+n}\cos n\psi + \frac{an}{m+n}\cos m\psi,$$

and $$y = \frac{am}{m+n}\sin n\psi + \frac{an}{m+n}\sin m\psi.$$

This envelope is an epicycloid or a hypocycloid according as m and n have the same or opposite signs. The vertices are in both cases on the given circle, and the cusps are on the circle whose radius is $\dfrac{m-n}{m+n}a$.

4. Find the envelope of a line which revolves uniformly about a point which moves uniformly in a fixed straight line.

Taking the fixed line as the axis of y, and, for the origin, the position which the moving point occupies when the revolving line coincides with the fixed line, the equation of the revolving line is

$$x = \tan\alpha\,(y - a\alpha);$$

whence $\quad x = a\sin^2\alpha \quad$ and $\quad y = a\sin\alpha\cos\alpha + a\alpha,$

or introducing in the double angle

$$x = \frac{a}{2}(1 - \cos 2\alpha), \qquad y = \frac{a}{2}(2\alpha + \sin 2\alpha).$$

The curve is therefore a cycloid referred to its vertex. (See Art. 290.)

5. Find the equation of the evolute of the parabola, using the equation of the normal in terms of its direction-ratio ; viz.,

$$y = mx - 2am - am^3.$$

$$27ay^2 = 4(x - 2a)^3.$$

6. Find the equation of the evolute of the cycloid by means of the equation of the normal in terms of ψ.

The equation of the normal is

$$x + y\frac{\sin\psi}{1 - \cos\psi} - a\psi = 0.$$

The equations of the evolute are

$$x = a(\psi + \sin\psi) \qquad \text{and} \qquad y = -a(1 - \cos\psi).$$

(Compare Art. 357.)

7. Show that the equation of the normal to the curve

$$x = a\cos\alpha\,(1 + \sin^2\alpha), \qquad y = a\sin\alpha\cos^2\alpha$$

is

$$x\sin\alpha + y\cos\alpha = a\sin 2\alpha;$$

and thence prove that the evolute is the four-cusped hypocycloid.

8. Find the equation of the evolute of the hypocycloid. Art. 297.

The equation of the normal is

$$x\cos\frac{a-2b}{2b}\psi - y\sin\frac{a-2b}{2b}\psi = a\cos\frac{a}{2b}\psi;$$

whence

$$x = a\left\{\frac{a-b}{a-2b}\cos\psi - \frac{b}{a-2b}\cos\frac{a-b}{b}\psi\right\},$$

$$y = a\left\{\frac{a-b}{a-2b}\sin\psi + \frac{b}{a-2b}\sin\frac{a-b}{b}\psi\right\}.$$

9. Determine the negative pedal (see Art. 381) of the parabola

$$y^2 = 4ax.$$

The semicubical parabola $(x - 4a)^3 = 27ay^2$

10. Determine the negative pedal of the cissoid

$$r = 2a\frac{\sin^2\theta}{\cos\theta}.$$

$$y^2 = -8ax.$$

11. Prove that the negative pedal of the spiral of Archimedes is the involute of the circle.

12. Determine the negative pedal of the curve

$$r = b\sin m\theta.$$

$$x = b\sin m\theta\cos\theta - mb\cos m\theta\sin\theta:$$
$$y = b\sin m\theta\sin\theta + mb\cos m\theta\cos\theta.$$

By putting for b, $\frac{m+1}{2}b - \frac{m-1}{2}b$, and for mb, $\frac{m+1}{2}b + \frac{m-1}{2}b$, it may be shown that this curve is a hypocycloid.

13. Derive the polar equation of the negative pedal of the curve

$$r^m = a^m\cos m\theta.$$

The rectangular equations are

$$x = a(\cos m\theta)^{\frac{1-m}{m}}\cos(1-m)\theta,\ y = a(\cos m\theta)^{\frac{1-m}{m}}\sin(1-m)\theta.$$

Whence, denoting the polar coordinates of the pedal by r' and θ',

$$\tan\theta' = \frac{y}{x} = \tan(1-m)\theta, \qquad \text{and} \qquad r' = a(\cos m\theta)^{\frac{1-m}{m}}.$$

therefore, eliminating θ,

$$r'^{\frac{m}{1-m}} = a^{\frac{m}{1-m}}\cos\frac{m\theta'}{1-m}.$$

14. Find the reciprocal of the parabola referred to its vertex.

Equation (1), Art. 385, is

$$xx' + yy' = k^2,$$

and treating the reciprocal as an envelope, the relation between the parameters is in this case

$$y'^2 = 4ax'.$$

Hence the reciprocal curve is $$ay^2 = -k^2x.$$

15. Determine the reciprocal of the circle

$$(x' - b)^2 + y'^2 = a^2.$$

The result is $$a^2(x^2 + y^2) = (k^2 - bx)^2,$$

the equation of a conic referred to its focus and axis. This conic is an ellipse, an hyperbola, or a parabola, according as the origin is within, without, or on the circumference of the circle.

16. Find the caustic by reflection for the cycloid, the incident rays being perpendicular to the base.

The inclination of the reflected ray is $\frac{1}{2}\pi - \psi$, and its equation is

$$y \sin\psi - x \cos\psi = a(\sin\psi - \psi\cos\psi);$$

whence $x = a(\psi - \sin\psi\cos\psi)$, and $y = a\sin^2\psi$:

or, introducing the double-angle,

$$x = \tfrac{1}{2}a(2\psi - \sin 2\psi) \qquad \text{and} \qquad y = \tfrac{1}{2}a(1 - \cos 2\psi).$$

The caustic is therefore a cycloid.

17. Find the caustic by reflection for parallel rays, the curve being a circle.

Putting the origin at the centre, and the axis of x parallel to the

incident rays, the coordinates of the point of incidence are $a \cos \theta$ and $a \sin \theta$, and the equation of the reflected ray is

$$y \cot 2\theta - x + \tfrac{1}{2}a \sec \theta = 0;$$

whence $\quad y = a \sin^3 \theta, \quad$ and $\quad x = \tfrac{1}{2}a \cos \theta \, (3 - 2 \cos^2 \theta).$

These are the equations of a two-cusped epicycloid, the radius of the fixed circle being $\tfrac{1}{2}a$, and that of the rolling circle $\tfrac{1}{4}a$.

18. Find the caustic by reflection of the parabola, the incident rays being perpendicular to the axis of the curve.

Taking the origin at the focus, the equation of the parabola is

$$y'^2 = 4a\,(x' + a);$$

and, denoting by ϕ the inclination of the normal to the negative direction of the axis of x, we have

$$y' = 2a \tan \phi \qquad \text{and} \qquad x' = a\,(\tan^2 \phi - 1).$$

The equation of the reflected ray is

$$y - y' = \tan\,(\tfrac{1}{2}\pi - 2\phi)\,(x - x') = \cot 2\phi\,(x - x'),$$

which reduces to

$$y \sin 2\phi - x \cos 2\phi = a \sec^2 \phi.$$

$$\therefore \qquad x = -a \sec^3 \phi \cos 3\phi \qquad \text{and} \qquad y = a \sec^3 \phi \sin 3\phi;$$

therefore the polar equation of the caustic is

$$r^{\frac{1}{3}} = a^{\frac{1}{3}} \sec \tfrac{1}{3}\pi - \theta.$$

CHAPTER XI.

Functions of Two or More Variables.

XXXVIII.

The Derivative Regarded as the Limit of a Ratio.

389. The difference between two values of a variable is frequently expressed by prefixing the symbol Δ to the symbol denoting the variable, and the difference between corresponding values of any function of the variable, by prefixing Δ to the symbol denoting the function. Hence x and $x + \Delta x$ denote two values of the independent variable, and $\Delta f(x)$ denotes the difference between the corresponding values of $f(x)$; that is, if $y = f(x)$,

$$\Delta y = \Delta f(x) = f(x + \Delta x) - f(x). \quad . \quad . \quad . \quad (1)$$

If we put $\Delta x = 0$, we shall have $\Delta y = 0$;

hence the ratio $$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

takes the indeterminate form $\frac{0}{0}$ when $\Delta x = 0$. The value assumed in this case is called *the limiting value of the ratio of the increments*, Δy and Δx, when the absolute values of these increments are diminished indefinitely.

390. To determine this limiting value, for a particular value a of x, we put a for x and z for Δx in the second member of

equation (2), and evaluate for $z = 0$, by the ordinary process (see Art. 96). Thus

$$\left.\frac{f(a+z)-f(a)}{z}\right]_0 = f'(a). \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

Therefore when Δx is diminished indefinitely, the limiting value of $\frac{\Delta y}{\Delta x}$ corresponding to $x = a$ is $\left.\frac{dy}{dx}\right]_a$, and, since a denotes any value of x, we have in general

$$limit\ of\ \frac{\Delta y}{\Delta x} = \frac{dy}{dx}.$$

If we denote by e the difference between the values of $\frac{\Delta y}{\Delta x}$ and $\frac{dy}{dx}$, we shall have

$$\frac{\Delta y}{\Delta x} = \frac{dy}{dx} + e, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

and the result established in the preceding article may be expressed thus—

$$e = 0 \qquad \text{when} \qquad \Delta x = 0;$$

in other words, *e is a quantity that vanishes with Δx.*

Partial Derivatives.

391. Let $$u = f(x, y),$$

in which x and y are two independent variables. The derivative of u with reference to x, y being regarded as constant, is denoted by $\frac{d}{dx}u$, and the derivative of u with reference to y, x being constant, by $\frac{d}{dy}u$. These derivatives are called *the partial* derivatives of u with reference to x and y respectively.

Adopting this notation, the result established in Art. 64 may be expressed thus:

$$du = \frac{d}{dx}u \cdot dx + \frac{d}{dy}u \cdot dy;$$

provided u *denotes a function that can be expressed by means of the elementary functions differentiated in Chapters* II *and* III.

It is now to be proved that this result is universally true.

392. Let $\Delta_x u$ denote the increment of u corresponding to Δx, y being unchanged, $\Delta_y u$ the increment corresponding to Δy, x being unchanged, and Δu the increment which u receives when x and y receive the simultaneous increments Δx and Δy. Let

$$u' = f(x + \Delta x, y),$$

and

$$u'' = f(x + \Delta x, y + \Delta y);$$

then

$$\Delta_x u = u' - u,$$

$$\Delta_y u' = u'' - u',$$

and

$$\Delta u = u'' - u;$$

hence

$$\Delta u = \Delta_x u + \Delta_y u'. \qquad (1)$$

Denoting by Δt the interval of time in which x, y, and u receive the increments Δx, Δy, and Δu, we have

$$\frac{\Delta u}{\Delta t} = \frac{\Delta_x u}{\Delta t} + \frac{\Delta_y u'}{\Delta t}. \qquad (2)$$

Since Δu is the actual increment of u in the interval Δt, the limit of the first member of equation (2) is, by Art. 390, $\frac{du}{dt}$, the rate of u. The limit of $\frac{\Delta_x u}{\Delta t}$ is the rate which u would have

were x the only variable; and, since $\frac{d}{dx}u \cdot dx$ is the value which du assumes when this supposition is made, if we put

$$\frac{d}{dx}u \cdot dx = d_x u,$$

this rate will be denoted by $\frac{d_x u}{dt}$. Hence by equation (4), Art. 390, equation (2) becomes

$$\frac{du}{dt} + e = \frac{d_x u}{dt} + e' + \frac{d_y u'}{dt} + e'',$$

in which e, e', and e'' vanish with Δt; but when $\Delta t = 0$, $\Delta x = 0$, and therefore $u' = u$; hence, putting $\Delta t = 0$, we have

$$\frac{du}{dt} = \frac{d_x u}{dt} + \frac{d_y u}{dt}.$$

Therefore $$du = d_x u + d_y u;$$

that is, $$du = \frac{d}{dx}u \cdot dx + \frac{d}{dy}u \cdot dy.$$

393. This result is usually written in the form

$$du = \frac{du}{dx}dx + \frac{du}{dy}dy,$$

but when written in this form it must be remembered that the fractions in the second member represent *partial derivatives*, the symbol du in the numerators standing for the quantities denoted above by $d_x u$ and $d_y u$, which are sometimes called *partial differentials*. The du that appears in the first member is called the *total differential* of u when x and y are both variable.

For example, if u is an implicit function of x and y satisfying the relation

$$u = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and it is required to express $\frac{dy}{dx}$ in terms of the derivatives of u; we have, by differentiating equation (1),

$$\frac{du}{dx}dx + \frac{du}{dy}dy = 0;$$

whence
$$\frac{dy}{dx} = -\frac{\frac{du}{dx}}{\frac{du}{dy}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This equation seems to be inconsistent with the principles of algebra; but when we remember that the du in the numerator is in reality $d_x u$, while that in the denominator is $d_y u$, the difficulty vanishes, since we have from equation (1)

$$du = d_x u + d_y u = 0; \qquad \text{or} \qquad d_x u = -d_y u.$$

394. The result deduced in Art. 392 is readily extended to the case of more than two independent variables.

Thus if
$$u = f(x, y, z, \cdot \cdot \cdot),$$

u', u'', etc. being defined as in Art. 392, it may be shown that

$$\Delta u = \Delta_x u + \Delta_y u' + \Delta_z u'' + \cdot \cdot \cdot,$$

and thence that

$$du = \frac{d}{dx}u \cdot dx + \frac{d}{dy}u \cdot dy + \frac{d}{dz}u \cdot dz + \ldots$$

The Approximate Values of Errors due to Small Errors of Observation.

395. The principle embodied in equation (4), Art. 390, is useful in determining approximately the relation between the

errors in any observed quantities, and the resulting error in a quantity derived from them by computation.

To illustrate, let the data be the two sides a and b and the included angle C of a plane triangle, and let it be required to determine the third side c.

The relation between c and the given parts is

$$c^2 = a^2 + b^2 - 2ab \cos C, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and it is required to ascertain the effect of certain small errors in the values of a, b, and C upon the resulting value of c.

Let Δa denote the error in the value of a, and Δc the corresponding change in the value of c; that is, the change produced by changing a to $a + \Delta a$ while b and C remain unchanged. Now, c being regarded as a function of a, we have, by Art. 390,

$$\frac{\Delta c}{\Delta a} = \frac{dc}{da} + e.$$

By differentiating equation (1), a and c being regarded as the only variables, we have

$$c\,dc = (a - b \cos C)\,da.$$

Hence approximately, when Δa is small,

$$\frac{\Delta c}{\Delta a} = \frac{a - b \cos C}{c},$$

or, since
$$a = c \cos B + b \cos C,$$

$$\Delta c = \cos B \cdot \Delta a.$$

396. To indicate that this expression is the approximate value of the error due to the error in a, we use the notation adopted in Art. 392: thus

$$\Delta_a c = \cos B \cdot \Delta a. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In a similar manner we obtain

$$\Delta_b c = \cos A \Delta b, \qquad \text{and} \qquad \Delta_C c = a \sin B \Delta C.$$

If we apply successively the increments Δa, Δb, and ΔC, c will receive three increments; of these the second and the third will, when Δb and ΔC are small, differ very little from $\Delta_b c$ and $\Delta_C c$ respectively. Hence the total error will differ very little from the sum of the partial errors whose approximate values are given above; therefore we have, approximately,

$$\Delta c = \cos B \Delta a + \cos A \Delta b + a \sin B \Delta C. \quad . \quad . \quad . \quad . \quad (2)$$

It is obvious that this result may be obtained by substituting the symbol Δ for d in the expression for the total differential of c regarded as a function of a, b, and C; viz.,

$$dc = \cos B\, da + \cos A\, db + a \sin B\, dC.$$

In applying equation (2), it must be remembered that ΔC is expressed in circular measure.

397. It should be remarked that the expressions for partial derivatives which involve the parts of a triangle depend not only upon the parts whose differentials are compared, but also upon the other two parts which appear among the given quantities. Thus, we have from equation (3) of the preceding article,

$$\frac{dc}{db} = \cos A,$$

the other parts involved being a and C.

But, if c is computed from the data b, A, and C, we have

$$c \sin (A + C) = b \sin C;$$

whence
$$\frac{dc}{db} = \frac{\sin C}{\sin (A + C)}.$$

The dc in the numerator of each of these expressions might be denoted by $d_b c$; the difference in value is dependent on the fact, that in the former case the total differential dc is assumed to be decomposed thus—

$$dc = d_b c + d_a c + d_C c;$$

while in the latter case we assume

$$dc = d_b c + d_A c + d_C c,$$

the partial differentials having radically different meanings in the two cases.

Examples XXXVIII.

1. Given $u = (x^2 + y^2)^{\frac{1}{3}}$, prove that

$$x\frac{du}{dx} + y\frac{du}{dy} = \tfrac{2}{3}u.$$

2. Given $u = \dfrac{xy}{x+y}$, prove that

$$x\frac{du}{dx} + y\frac{du}{dy} = u.$$

3. Given $u = \tan^{-1}\left(\dfrac{x-y}{x+y}\right)^{\frac{1}{2}}$, prove that

$$x\frac{du}{dx} + y\frac{du}{dy} = 0.$$

4. Given $u = \log_y x$, to find $\dfrac{du}{dx}$ and $\dfrac{du}{dy}$.

$$\frac{du}{dx} = \frac{1}{x\log y}; \quad \frac{du}{dy} = \frac{-\log x}{y(\log y)^2}.$$

5. Given $u = \log[x + \sqrt{(x^2 + y^2)}]$, prove that

$$\left(x\frac{d}{dx} + y\frac{d}{dy}\right)u = 1.$$

6. Given $u = \log(x^3 + y^3 + z^3 - 3xyz)$, prove that

$$\frac{du}{dx} + \frac{du}{dy} + \frac{du}{dz} = \frac{3}{x + y + z}.$$

7. In a plane triangle, determine the approximate value of ΔC when the data are the three sides.

$$\Delta C = \frac{\Delta c - \cos B \Delta a - \cos A \Delta b}{b \sin A}.$$

8. Find the approximate value of Δc when the data are the base line b and the adjacent angles A and C.

$$\Delta c = a \operatorname{cosec} B \Delta C - c \cot B \Delta A + \frac{c\Delta b}{b}.$$

9. Find the approximate value of ΔC when the four parts involved are A, C, a, and c.

$$\Delta C = \frac{\sin A}{a \cos C}\Delta c + \frac{\tan C}{\tan A}\Delta A - \frac{\tan C}{a}\Delta a.$$

10. In a plane triangle, given A, B, and c, determine p (the perpendicular on c), and find the partial derivative of p with reference to A.

$$p = c\frac{\sin A \sin B}{\sin(A+B)}, \quad \text{whence} \quad \frac{dp}{dA} = \frac{c \sin^2 B}{\sin^2(A+B)}.$$

11. In a plane triangle, given A, B, and p, find the partial derivative of c with reference to A.

$$\frac{dc}{dA} = p\frac{\cos(A+B)}{\sin A \sin B} - c \cot A.$$

12. The area k of a plane triangle being determined from two sides and the included angle, prove that

$$\Delta k = \tfrac{1}{2}(b \sin C \Delta a + a \sin C \Delta b + ab \cos C \Delta C).$$

13. The area k of a plane triangle being determined from the three sides, prove that

$$\frac{\Delta k}{\Delta a} = [b^2 + c^2 - a^2]\frac{a}{8k} = \frac{a \cot A}{2}.$$

14. In a right spherical triangle, given $\sin a = \sin A \sin c$, derive

$$\cot a \Delta a = \cot A \Delta A + \cot c \Delta c.$$

Take logarithmic derivatives.

15. In a right spherical triangle, given $\cos c = \cot A \cot B$, derive

$$\frac{\Delta c}{2 \cot c} = \frac{\Delta A}{\sin 2A} + \frac{\Delta B}{\sin 2B}.$$

16. Determine the relation between the errors when the parts involved are the three sides and one angle of a spherical triangle.

$$\Delta c = \cos B \Delta a + \cos A \Delta b + \sin b \sin A \Delta C.$$

XXXIX.

The Second and Higher Derivatives regarded as Limits.

398. In Art. 390 it is shown that

$$\frac{\Delta y}{\Delta x} = \frac{dy}{dx} + e.$$

In this equation e is a function of x and likewise of Δx; hence the derivative $\frac{de}{dx}$ is in general a function of x and of Δx. It is

also proved in the same article that e becomes zero when Δx vanishes; that is, *e assumes a constant value independent of the value of* x *when* Δx *becomes zero*; hence, when Δx is zero, *the derivative* of e with reference to x must take the value zero, whatever be the value of x; in other words,

$$\frac{de}{dx} \text{ vanishes with } \Delta x.$$

In a similar manner it may be shown that each of the higher derivatives of e with reference to x vanishes when $\Delta x = 0$.

399. Since $\frac{\Delta y}{\Delta x}$ is a function of x, $\Delta \frac{\Delta y}{\Delta x}$ will denote the increment of this function corresponding to Δx. Employing the symbol $\frac{\Delta}{\Delta x}$ to denote the operation of taking this increment, and dividing the result by Δx, we obtain, by applying to this function the principle expressed in equation (4), Art. 390,

$$\frac{\Delta}{\Delta x} \cdot \frac{\Delta y}{\Delta x} = \frac{d}{dx} \cdot \frac{\Delta y}{\Delta x} + e', \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$= \frac{d}{dx}\left(\frac{dy}{dx} + e\right) + e'$$

$$= \frac{d^2y}{dx^2} + \frac{de}{dx} + e'.$$

In this equation both e' and $\frac{de}{dx}$ vanish with Δx by the preceding article; hence the sum of these quantities likewise vanishes with Δx, and may be denoted by e. Thus we write

$$\frac{\Delta}{\Delta x} \cdot \frac{\Delta y}{\Delta x} = \frac{d^2y}{dx^2} + e. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

400. Since Δx is an arbitrary quantity it may be regarded as constant, whence $\Delta \dfrac{\Delta y}{\Delta x}$ is the increment of a fraction whose denominator is constant; but this is evidently equivalent to the result obtained by dividing the increment of the numerator by the denominator; that is,

$$\Delta \frac{\Delta y}{\Delta x} = \frac{\Delta \cdot \Delta y}{\Delta x}.$$

The numerator $\Delta \cdot \Delta y$ is usually denoted by the symbol $\Delta^2 y$; hence equation (2) may be written thus:

$$\frac{\Delta^2 y}{\Delta x^2} = \frac{d^2 y}{dx^2} + e, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and, since e vanishes with Δx, it follows that the second derivative is the limit of the expression in the first member of equation (3).

In a similar manner it may be shown that each of the higher derivatives is the limit of the expression obtained by substituting Δ for d in the symbol denoting the derivative.

Higher Partial Derivatives.

401. The partial derivatives of u with reference to x and y are themselves functions of x and y. Their partial derivatives, viz.,

$$\frac{d}{dx}\cdot\frac{du}{dx}, \quad \frac{d}{dy}\cdot\frac{du}{dx}, \quad \frac{d}{dx}\cdot\frac{du}{dy}, \quad \text{and} \quad \frac{d}{dy}\cdot\frac{du}{dy},$$

are called partial derivatives of u of the *second order*.

It will now be shown that the second and third of these derivatives, although results of different operations, are in fact identical; that is, that

$$\frac{d}{dy}\cdot\frac{du}{dx} = \frac{d}{dx}\cdot\frac{du}{dy}.$$

Employing the notation introduced in Art. 392, we have

$$\Delta_x u = f(x + \Delta x, y) - f(x, y);$$

if in this equation we replace y by $y + \Delta y$, we obtain a new value of $\Delta_x u$, and, denoting this value by $\Delta'_x u$, we have

$$\Delta'_x u = f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y).$$

Since this change in the value of $\Delta_x u$ results from the increment received by y, the expression for the increment received by $\Delta_x u$ will be $\Delta_y(\Delta_x u)$; hence

$$\Delta_y(\Delta_x u) = \Delta'_x u - \Delta_x u,$$

or

$$\Delta_y(\Delta_x u) = f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y) - f(x+\Delta x, y) + f(x, y).$$

The value of $\Delta_x(\Delta_y u)$, obtained in a precisely similar manner, is identical with that just given; hence

$$\Delta_y(\Delta_x u) = \Delta_x(\Delta_y u). \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Since Δx is constant, we have, as in Art. 400,

$$\frac{\Delta_y(\Delta_x u)}{\Delta x} = \Delta_y \cdot \frac{\Delta_x u}{\Delta x}.$$

Hence, dividing both members of equation (1) by $\Delta x \cdot \Delta y$, we have

$$\frac{\Delta_y}{\Delta y} \cdot \frac{\Delta_x u}{\Delta x} = \frac{\Delta_x}{\Delta x} \cdot \frac{\Delta_y u}{\Delta y}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

or, employing the symbol $\dfrac{\Delta}{\Delta x}$ as in Art. 399,

$$\frac{\Delta}{\Delta y} \cdot \frac{\Delta}{\Delta x} u = \frac{\Delta}{\Delta x} \cdot \frac{\Delta}{\Delta y} u.$$

From this result, by a course of reasoning similar to that employed in Art. 399, we obtain

$$\frac{d}{dy} \cdot \frac{du}{dx} = \frac{d}{dx} \cdot \frac{du}{dy} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

402. The partial derivatives of the second order are usually denoted by

$$\frac{d^2u}{dx^2}, \qquad \frac{d^2u}{dx\,dy}, \qquad \frac{d^2u}{dy^2},$$

the factors dx and dy in the denominator of the second being, by virtue of formula (3), interchangeable, as in the case of an ordinary product.

The numerators of the above fractions are of course not identical. Compare Art. 393.

Formula (3) of the preceding article is readily verified in any particular case. Thus, given

$$u = y^x,$$

whence $\quad \dfrac{du}{dx} = y^x \log y, \quad$ and $\quad \dfrac{du}{dy} = xy^{x-1};$

$$\therefore \qquad \frac{d}{dy} \cdot \frac{du}{dx} = y^{x-1}(x \log y + 1) = \frac{d}{dx} \cdot \frac{du}{dy}.$$

403. Equation (3) of Art. 401 shows that a differential expression of the form

$$Mdx + Ndy,$$

in which M and N are functions of x and y, does not always form the total differential of a function of x and y. For if we assume the existence of a function u fulfilling the condition,

$$du = Mdx + Ndy, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

we shall have $\quad M = \frac{du}{dx} \quad$ and $\quad N = \frac{du}{dy}$;

hence, by the equation cited above, we must have

$$\frac{dM}{dy} = \frac{dN}{dx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

Equation (2) constitutes therefore a necessary condition in order that the expression $Mdx + Ndy$ may be an *exact differential.*

404. The theorem proved in Art. 401 may be expressed thus—*the operations of taking the derivative with respect to two independent variables are commutative;* that is, they may be interchanged without affecting the result obtained.

This theorem may be extended to derivatives higher than the second, and also to functions of more than two independent variables. For it has been proved that we may, without affecting the result, interchange any two consecutive differentiations, and it is obvious that, by successive interchanges of consecutive differentiations, we can alter the order of differentiation in any manner desired. Hence all differentiations with respect to independent variables are commutative.

In accordance with this theorem, the result of differentiating m times with respect to x, n times with respect to y, and p times with respect to z, may, without regard to the order of differentiation, be expressed by the symbol

$$\frac{d^{m+n+p}u}{dx^m dy^n dz^p}.$$

Symbols of Operation.

405. The symbol $\frac{d}{dx}$ has already been employed to denote the operation of taking the derivative, with respect to x, of the function to which it is prefixed, this derivative being a

partial derivative when the quantity is regarded as a function of more than one variable. So likewise the compound symbol $\frac{d}{dy}\cdot\frac{d}{dx}$ indicates the operation of taking the derivative with reference to x, and the derivative of the result with reference to y; the symbol written last being applied first, since the compound symbol is prefixed to the function. It has however been shown in Art. 401 that in this case the operations are commutative; that is, that

$$\frac{d}{dx}\cdot\frac{d}{dy}=\frac{d}{dy}\cdot\frac{d}{dx}.$$

The compound symbol is called the *symbolic product* of the simple symbols. In this case the product is commutative like an ordinary algebraic product.

The quantity affected by a symbol of operation is called the *operand*. When the symbolic notation is employed, it is to be understood that the product of all the factors following a symbol of operation constitutes the operand.

Commutative and Distributive Operations.

406. When m is constant we have

$$d\,(mu)=m\,du\,;$$

hence when u is a function of x

$$\frac{d}{dx}mu=m\frac{d}{dx}u. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This equation indicates that the operation of multiplying by a constant, and the operation of taking the derivative with reference to x, are commutative. It follows that the factors of the

symbolic product $y\frac{d}{dx}$, when y is a variable independent of x, are commutative; for, in performing the operation $\frac{d}{dx}$, y is regarded as a constant.

On the other hand, $x\frac{d}{dx}$ is not commutative; for

$$\frac{d}{dx}xu = x\frac{du}{dx} + u,$$

while
$$x\frac{d}{dx}u = x\frac{du}{dx}.$$

407. A repeated application of the same symbol, whether simple or compound, is indicated by affixing an index to the symbol: thus—

$$\left(y\frac{d}{dx}\right)^2 = y\frac{d}{dx}\cdot y\frac{d}{dx},$$

and, since the operations indicated by the symbol are commutative, we have

$$\left(y\frac{d}{dx}\right)^2 = y^2\frac{d^2}{dx^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

On the other hand, the operations indicated by the symbol $x\frac{d}{dx}$ not being commutative, $\left(x\frac{d}{dx}\right)^2$ is *not* equal to $x^2\frac{d^2}{dx^2}$.

The result obtained by adding or subtracting the results arising from the application of two operative symbols may be expressed by connecting the symbols by the appropriate sign: thus—

$$\left(\frac{d}{dx} + m\right)u = \frac{du}{dx} + mu. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Since the sum expressed in the second member of this equation is commutative, the symbolic sum expressed in the first member is likewise commutative.

408. When u and v are functions of x we have

$$\frac{d}{dx}(u+v) = \frac{d}{dx}u + \frac{d}{dx}v. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This equation signifies that the operation of taking a derivative, as applied to a sum, is a *distributive* operation; that is, the result which arises from performing this operation upon a sum is identical with the sum of the results obtained by performing the operation upon each quantity separately. Since this distributive principle is applicable to ordinary algebraic multiplication, the first member of equation (1) is expanded exactly as it would be, if the symbols represented algebraic quantities.

Equation (1) Art. 407 expresses the fact that the application of an exponent to a symbolic product of the form $y\frac{d}{dx}$ is distributive; and it is obvious from the mode in which this equation arises that *an exponent is distributive whenever the symbolic product to which it is applied is commutative.*

Symbolic Transformations.

409. The formulas of algebraic expansion are consequences of the commutative and distributive nature of algebraic multiplication; hence it follows that a symbolic product or power may be expanded by these formulas; provided all the factors involved represent commutative operations. Thus—

$$\left(\frac{d}{dx}+a\right)\left(\frac{d}{dx}-a\right)u = \frac{d^2u}{dx^2} - a^2u.$$

Again the total differential of a function of x and y is expressed by the equation

$$du = dx\frac{du}{dx} + dy\frac{du}{dy} = \left(dx\frac{d}{dx} + dy\frac{d}{dy}\right)u,$$

in which all the factors have commutative products, since dx is regarded as a constant in differentiating with reference to x as well as in differentiating with reference to y. Hence we have

$$d^n u = \left(dx\frac{d}{dx} + dy\frac{d}{dy}\right)^n u = dx^n\frac{d^n u}{dx^n} + n\,dx^{n-1}dy\frac{d^n u}{dx^{n-1}dy} + \cdots,$$

a formula giving *the* n*th total derivative* of a function of two variables.

410. The result deduced below is frequently employed in transforming operative symbols.

Let u denote a function of θ; then we have, by differentiation,

$$\frac{d}{d\theta}\varepsilon^{n\theta}u = \varepsilon^{n\theta}\left(\frac{d}{d\theta} + n\right)u,$$

and multiplying by $\varepsilon^{-n\theta}$

$$\varepsilon^{-n\theta}\frac{d}{d\theta}\varepsilon^{n\theta}\cdot u = \left(\frac{d}{d\theta} + n\right)u. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Applying now the symbols whose equivalence is expressed in this equation to the equation itself, we have

$$\varepsilon^{-n\theta}\frac{d}{d\theta}\varepsilon^{n\theta}\cdot\varepsilon^{-n\theta}\frac{d}{d\theta}\varepsilon^{n\theta}\cdot u = \varepsilon^{-n\theta}\frac{d^2}{d\theta^2}\varepsilon^{n\theta}u = \left(\frac{d}{d\theta} + n\right)^2\cdot u;$$

and, by repeating this process, we have in general

$$\varepsilon^{-n\theta}\frac{d^r}{d\theta^r}\varepsilon^{n\theta}\cdot u = \left(\frac{d}{d\theta} + n\right)^r\cdot u.* \quad . \quad . \quad . \quad . \quad (2)$$

* This equation is equivalent to the result obtained by means of Leibnitz' theorem in Art. 90.

Euler's Theorem concerning Homogeneous Functions.

411. A homogeneous algebraic function of the nth degree involving two variables may be put in the form

$$u = x^n f\left(\frac{y}{x}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

In this expression n admits of fractional and negative values, and moreover u may be a fraction whose numerator and denominator are homogeneous, n being in this case the difference between the numbers expressing their degrees. Thus—

$$u = \frac{\sqrt{x} + \sqrt{y}}{x(x+y)}$$

is a homogeneous function in which $n = -\frac{3}{2}$.

In equation (1) f may denote a transcendental function, and in this case u is still called a homogeneous function. It is to be noticed that when u is transcendental the expression under the functional sign f must be of the zero degree. Thus—

$$u = \log\left[x + \sqrt{(x^2 + y^2)}\right]$$

is *not* a homogeneous function because the quantity under the sign log, although homogeneous, is not of the zero degree.

412. By differentiating equation (1), we derive

$$\frac{du}{dx} = nx^{n-1} f\left(\frac{y}{x}\right) - x^{n-2} y f'\left(\frac{y}{x}\right),$$

and
$$\frac{du}{dy} = x^{n-1} f'\left(\frac{y}{x}\right);$$

whence
$$x\frac{du}{dx} + y\frac{du}{dy} = nx^n f\left(\frac{y}{x}\right) = nu.$$

or, symbolically, $$\left(x\frac{d}{dx}+y\frac{d}{dy}\right)u = nu, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

when u is a homogeneous function of the nth degree.

Again, the derivatives of u are homogeneous functions of the $(n-1)$th degree; hence, by the theorem expressed in equation (1), we have

$$\left(x\frac{d}{dx}+y\frac{d}{dy}\right)\frac{du}{dx} = (n-1)\frac{du}{dx}, \text{ and } \left(x\frac{d}{dx}+y\frac{d}{dy}\right)\frac{du}{dy} = (n-1)\frac{du}{dy};$$

whence, expanding,

$$x\frac{d^2u}{dx^2}+y\frac{d^2u}{dy\,dx} = (n-1)\frac{du}{dx}, \quad \text{and} \quad x\frac{d^2u}{dy\,dx}+y\frac{d^2u}{dy^2} = (n-1)\frac{du}{dy};$$

multiplying and adding,

$$x^2\frac{d^2u}{dx^2} + 2xy\frac{d^2u}{dx\,dy} + y^2\frac{d^2u}{dy^2} = (n-1)\left[x\frac{du}{dx}+y\frac{du}{dy}\right]$$

hence $$x^2\frac{d^2u}{dx^2} + 2xy\frac{d^2u}{dx\,dy} + y^2\frac{d^2u}{dy^2} = n(n-1)u. \quad . \quad . \quad (2)$$

The results expressed in equations (1) and (2) and similar results involving higher derivatives are known as *Euler's Theorems.*

Examples XXXIX.

1. Given $u = \sec(y+ax) + \tan(y-ax)$, prove that

$$\frac{d^2u}{dx^2} = a^2\frac{d^2u}{dy^2}.$$

2. Verify the theorem $\frac{d^2u}{dx\,dy} = \frac{d^2u}{dy\,dx}$ when $u = \sin(xy^2)$.

3. Verify the theorem $\frac{d^2u}{dx\,dy} = \frac{d^2u}{dy\,dx}$ when $u = \log\tan(ax+y^2)$.

4. Verify the theorem $\dfrac{d^2u}{dy\,dx} = \dfrac{d^2u}{dx\,dy}$ when $u = \tan^{-1}\dfrac{x}{y}$.

5. Verify the theorem $\dfrac{d^3u}{dy\,dx^2} = \dfrac{d^3u}{dx^2\,dy}$ when $u = y\log(1 + xy)$.

6. Given $\quad u = \sin x \cos y$, prove that

$$\frac{d^4u}{dy^2\,dx^2} = \frac{d^4u}{dx^2\,dy^2} = \frac{d^4u}{dx\,dy\,dx\,dy}.$$

7. Given $\quad u = x^3z^4 + e^xy^2z^3 + x^2y^2z^2$, derive

$$\frac{d^4u}{dx^2\,dy\,dz} = 6ye^xz^2 + 8yz.$$

8. Given $\quad u = \dfrac{1}{\sqrt{(4ab - c^2)}}$, prove that

$$\frac{d^2}{dc^2}u = \frac{d^2}{da\,db}u.$$

9. Given $\quad u = (x + y)^2$, prove that

$$x\frac{d^2u}{dx^2} + y\frac{d^2u}{dx\,dy} = \frac{du}{dx}.$$

10. Given $\quad u = \log(x^3 + y^3 + z^3 - 3xyz)$, prove that

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + 2\frac{d^2u}{dx\,dy} + 2\frac{d^2u}{dy\,dz} + 2\frac{d^2u}{dz\,dx} = -\frac{9}{(x + y + z)^2}.$$

Employ the symbol $\left(\dfrac{d}{dx} + \dfrac{d}{dy} + \dfrac{d}{dz}\right)^2 u$, *and see Ex.* XXXVIII, 6.

11. When $u = (x^2 + y^2)^{\frac{1}{2}}$, verify the formula

$$x^2\frac{d^2u}{dx^2} + 2xy\frac{d^2u}{dx\,dy} + y^2\frac{d^2u}{dy^2} = 0. \quad \textit{See Art. } 412.$$

12. When $u = (x^2 + y^2)^{\frac{1}{2}}$, verify the formula

$$x^2\frac{d^2u}{dx^2} + 2xy\frac{d^2u}{dx\,dy} + y^2\frac{d^2u}{dy^2} = \tfrac{3}{4}u.$$

13. Given $\qquad u = \dfrac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}}$, prove that

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} = 0.$$

14. Determine the value of

$$x\frac{du}{dx} + y\frac{du}{dy}, \qquad \text{when} \qquad u = \tan^{-1}\frac{x^2-y^2}{ax}.$$

Solution :—

Since tan u is a homogeneous function of the first degree, we have, by Euler's theorem,

$$\left(x\frac{du}{dx} + y\frac{du}{dy}\right)\sec^2 u = \tan u\,;$$

whence
$$x\frac{du}{dx} + y\frac{du}{dy} = \frac{ax\,(x^2-y^2)^2}{a^2x^2 + (x^2-y^2)^2}.$$

15. If $u = \sin v$, v being a homogeneous function of the nth degree, determine the value of $x\dfrac{du}{dx} + y\dfrac{du}{dy}$.

$$x\frac{du}{dx} + y\frac{du}{dy} = nv\cos v.$$

16. Given $\qquad u = \dfrac{1}{b}\,\varepsilon^{\frac{ac^2}{b^2}}$, prove that

$$\left(\frac{d}{db}\right)^2 u = \frac{d}{da}\left(\frac{d}{dc}\right)^2 u = \frac{u}{b^2}\left[2 + \frac{10ac^2}{b^2} + \frac{4a^2c^4}{b^4}\right].$$

17. Given $u = \epsilon^{ax+by}$, find the third total differential by the formula deduced in Art. 409.

$$d^3u = (a^3dx^3 + 3a^2b\,dx^2dy + 3ab^2dx\,dy^2 + b^3dy^3)\epsilon^{ax+by}.$$

18. Given $u = (x^2 + y^2)^{\frac{1}{2}}$, find the second total differential.

$$d^2u = (y^2dx^2 - 2xy\,dx\,dy + x^2dy^2)\,(x^2 + y^2)^{-\frac{3}{2}}.$$

19. Apply the symbol $a_0\dfrac{d}{da_1} + 2a_1\dfrac{d}{da_2} + 3a_2\dfrac{d}{da_3}$ to the expression

$$a_0^2a_3^2 - 6a_0a_1a_2a_3 + 4a_0a_2^3 + 4a_1^3a_3 - 3a_1^2a_2^2.$$

Result o.

20. Operate on $a_0a_2a_4 + 2a_1a_2a_3 - a_0a_3^2 - a_1^2a_4 - a_2^3$ with the symbol $a_0\dfrac{d}{da_1} + 2a_1\dfrac{d}{da_2} + 3a_2\dfrac{d}{da_3} + 4a_3\dfrac{d}{da_4}$.

Result o.

21. Show that the symbols $y\dfrac{d}{dx}$ and $a_0\dfrac{d}{da_1} + 2a_1\dfrac{d}{da_2} + 3a_2\dfrac{d}{da_3}$ give the same result when applied to the expression

$$(a_0a_2 - a_1^2)\,x^2 + (a_0a_3 - a_1a_2)\,xy + (a_1a_3 - a_2^2)\,y^2.$$

22. Prove that $8\dfrac{d}{dx}x^{\frac{3}{2}}\dfrac{d^2}{dx^2}\epsilon^{\sqrt{x}} = \epsilon^{\sqrt{x}}$.

23. Show that the result obtained by applying the symbol

$$\left(\frac{d}{dx}\right)^n x^{n+\frac{1}{2}}\left(\frac{d}{dx}\right)^{n+1} \quad \text{to} \quad \frac{x^{\frac{r}{2}}}{1\cdot 2\cdots r} \quad \text{is} \quad \frac{x^{\frac{r'}{2}}}{2^{2n+1}\cdot 1\cdot 2\cdots r'},$$

in which $r' = r - 2n - 1$.

24. Show that the result obtained by applying the symbol given in the preceding example to $\epsilon^{\sqrt{x}}$ is $\dfrac{\epsilon^{\sqrt{x}}}{2^{2n+1}}$.

Expanding the function by the exponential theorem, the application of the symbol to any term of the series produces a preceding term.

XL.

Change of the Independent Variable.

413. It is frequently desirable to transform expressions involving derivatives with reference to x into equivalent expressions, in which some variable connected with x by a known relation is the independent variable. This process is called *changing the independent variable.*

Let y denote the function whose derivatives occur in the given expression, and let θ denote the new independent variable, the relation between x and θ being, for example,

$$x = \tan\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

To obtain an expression for the first derivative it is only necessary to substitute the value of dx derived from equation (1), viz.,

$$dx = \sec^2\theta\, d\theta; \qquad \text{whence} \qquad \frac{dy}{dx} = \cos^2\theta\frac{dy}{d\theta}. \quad . \quad . \quad (2)$$

414. In the case of $\dfrac{d^2y}{dx^2}$ we cannot substitute the value of dx^2; because the d^2y in the numerator of this expression denotes the value which this differential assumes when dx is constant, while the d^2y in $\dfrac{d^2y}{d\theta^2}$ denotes the value assumed when $d\theta$ is constant (see Art. 80). We must therefore differentiate the expression for the first derivative and divide by the value of dx: thus from equation (2) we obtain

$$\frac{d^2y}{dx^2} = \frac{d}{\sec^2\theta\, d\theta}\left[\cos^2\theta\frac{dy}{d\theta}\right] =$$

$$\cos^2\theta\left[\cos^2\theta\frac{d^2y}{d\theta^2} - 2\cos\theta\sin\theta\frac{dy}{d\theta}\right] \quad . \quad . \quad (3)$$

In like manner an expression for the third derivative may be obtained.

415. As an application let it be required to transform the differential equation

$$\frac{d^2y}{dx^2} + \frac{2x}{1 + x^2}\frac{dy}{dx} + \frac{y}{(1 + x^2)^2} = 0$$

into one in which θ is the independent variable, the given relation being $x = \tan\theta$.

Eliminating x, $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ by means of equations (1), (2), and (3), we have

$$\cos^4\theta\frac{d^2y}{d\theta^2} - 2\cos^3\theta\sin\theta\frac{dy}{d\theta} + \frac{2\tan\theta}{\sec^2\theta}\cos^2\theta\frac{dy}{d\theta} + \frac{y}{\sec^4\theta} = 0;$$

whence, reducing we have

$$\frac{d^2y}{d\theta^2} + y = 0.$$

416. Expressions involving derivatives of y with reference to x may be transformed into equivalent expressions involving derivatives of x with reference to y. In this case we have

$$\frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}} \quad \therefore \quad \frac{d^2y}{dx^2} = -\frac{\dfrac{d^2x}{dy^2}\dfrac{dy}{dx}}{\left(\dfrac{dx}{dy}\right)^2} = -\frac{\dfrac{d^2x}{dy^2}}{\left(\dfrac{dx}{dy}\right)^3}:$$

For example, by means of these substitutions the expression

$$\rho = \frac{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}} \qquad \text{becomes} \qquad \rho = -\frac{\left[1 + \left(\dfrac{dx}{dy}\right)^2\right]^{\frac{3}{2}}}{\dfrac{d^2x}{dy^2}}.$$

Transformations of Certain Operative Symbols.

417. If we put $x = \varepsilon^\theta$, we have

$$dx = \varepsilon^\theta\, d\theta = x\, d\theta;$$

whence
$$x\frac{d}{dx} = \frac{d}{d\theta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

By differentiation we obtain

$$\frac{d}{dx}\, x^r \cdot u = x^{r-1}\left[x\frac{d}{dx} + r\right] \cdot u;$$

whence by equation (1)

$$x^{1-r}\frac{d}{dx}\, x^r \cdot u = \left[\frac{d}{d\theta} + r\right]u; \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

that is, the symbols $x^{1-r}\dfrac{d}{dx}\, x^r$ and $\left[\dfrac{d}{d\theta} + r\right]$ are equivalent.

Now, putting $r - 1$ for r in these symbols, and applying them to equation (2), we have

$$x^{2-r}\frac{d}{dx}\, x^{r-1} \cdot x^{1-r}\frac{d}{dx}\, x^r \cdot u = x^{2-r}\frac{d^2}{dx^2}\, x^r \cdot u =$$
$$\left[\frac{d}{d\theta} + r\right]\left[\frac{d}{d\theta} + r - 1\right]u^r;$$

and, by repeated applications of this process, we have in general

$$x^{n-r}\frac{d^n}{dx^n}\, x^r \cdot u = \left[\frac{d}{d\theta} + r\right]\left[\frac{d}{d\theta} + r - 1\right]\left[\frac{d}{d\theta} + r - 2\right]\cdots$$
$$\left[\frac{d}{d\theta} + r - n + 1\right]u. \quad . \qquad (3)$$

In this equation r admits of negative and fractional values.

If we put $r = 0$ in equation (3), we have

$$x^n \frac{d^n}{dx^n} u = \frac{d}{d\theta}\left[\frac{d}{d\theta} - 1\right]\left[\frac{d}{d\theta} - 2\right]\cdots\left[\frac{d}{d\theta} - n + 1\right] u. \quad . \; . \; (4)$$

and if we put $r = n$ in the same equation

$$\frac{d^n}{dx^n} x^n \cdot u = \left[\frac{d}{d\theta} + n\right]\left[\frac{d}{d\theta} + n - 1\right]\cdots\cdot\left[\frac{d}{d\theta} + 1\right] u. \quad . \; . \; (5)$$

418. Any two compound symbols of the forms occurring in the second members of the equations in the preceding article are commutative, since they involve differentiation and multiplication by constants only; therefore any two symbols of the forms given in the first members are likewise commutative. For example, the symbol $\frac{d^m}{dx^m} x^{m+p} \frac{d^p}{dx^p}$ admits of separation into two factors having the forms given in equations (5) and (4), respectively; hence, commuting these factors, we have

$$\frac{d^m}{dx^m} x^{m+p} \frac{d^p}{dx^p} u = x^p \frac{d^{m+p}}{dx^{m+p}} x^m \cdot u,$$

the common value of these symbols being

$$\left[\frac{d}{d\theta} + m\right]\left[\frac{d}{d\theta} + m - 1\right]\cdots\left[\frac{d}{d\theta} - p + 1\right]$$

The essential characteristic of commutative factors of the form here considered is that the sum of the exponents equals in each case the sum of the indices of the derivatives.

Expressions Involving Partial Derivatives.

419. Let u denote a function of x and y, and let r and θ be two new independent variables connected with x and y by two

given equations. It is required to express the partial derivatives of u with reference to x and y in terms of derivatives with reference to r and θ.

Now, since in this case u may be regarded as a function of r and θ, we have, by the theorem of Art. 392,

$$du = \frac{du}{dr}\,dr + \frac{du}{d\theta}\,d\theta\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In accordance with the same theorem, if we suppose y to be constant du in equation (1) will become $d_x u$ which is the numerator of the partial derivative $\frac{du}{dx}$. Upon the same supposition dr and $d\theta$ become $d_x r$ and $d_x\theta$, while the ratios $\frac{du}{dr}$ and $\frac{du}{d\theta}$, being independent of the absolute values of the differentials involved, are not affected by the supposition. Hence dividing by dx, we have

$$\frac{du}{dx} = \frac{du}{dr}\frac{dr}{dx} + \frac{du}{d\theta}\frac{d\theta}{dx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In like manner, making x constant, we derive

$$\frac{du}{dy} = \frac{du}{dr}\frac{dr}{dy} + \frac{du}{d\theta}\frac{d\theta}{dy}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

420. Let us now assume the given relations to be

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

It is to be remembered that the four coefficients $\frac{dr}{dx}$, $\frac{d\theta}{dx}$, $\frac{dr}{dy}$, and $\frac{d\theta}{dy}$ are *the partial derivatives* of r and θ with reference to x and y; their values therefore are not to be obtained directly from equations (1), but from the expressions for r and θ in terms of x and y.

Thus, from equations (1), we obtain

$$r^2 = x^2 + y^2, \qquad \text{and} \qquad \theta = \tan^{-1}\frac{y}{x};$$

whence

$$\left.\begin{aligned} \frac{dr}{dx} &= \frac{x}{r} = \cos\theta & \qquad \frac{d\theta}{dx} &= -\frac{y}{r^2} = -\frac{\sin\theta}{r} \\ \frac{dr}{dy} &= \frac{y}{r} = \sin\theta & \qquad \frac{d\theta}{dy} &= \frac{x}{r^2} = \frac{\cos\theta}{r} \end{aligned}\right\} \quad . \ . \ . \ (3)$$

The above method of proceeding should be carefully noticed, since the values of the partial derivatives $\frac{dx}{dr}$, $\frac{dx}{d\theta}$, etc., which would be obtained by direct differentiation of equations (1) are *not* the reciprocals of the derivatives required; for we obviously cannot assume $\frac{d_x r}{dx} \cdot \frac{d_r x}{dr}$ equal to unity.*

Substituting in equations (2) and (3) of the preceding article the values of the coefficients given by equations (3), we have

$$\frac{du}{dx} = \frac{du}{dr}\cos\theta - \frac{du}{d\theta}\frac{\sin\theta}{r}. \quad . \ . \ . \ . \ . \ . \quad (4)$$

$$\frac{du}{dy} = \frac{du}{dr}\sin\theta + \frac{du}{d\theta}\frac{\cos\theta}{r}. \quad . \ . \ . \ . \ . \ . \quad (5)$$

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* The values of $\frac{dr}{dx}$ and $\frac{d\theta}{dx}$ may also be found by elimination between the derivatives of the given equations with reference to x; viz.,

$$1 = \cos\theta\,\frac{dr}{dx} - r\sin\theta\,\frac{d\theta}{dx},$$

and

$$0 = \sin\theta\,\frac{dr}{dx} + r\cos\theta\,\frac{d\theta}{dx}.$$

When, however, the given equations can be solved for the new variables, the process given in the text is preferable.

421. Expressions for the higher derivatives may be obtained by differentiating equations (4) and (5). For example, let it be required to transform the expression

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2}$$

by making r and θ the independent variables.

In differentiating equation (4), it must be noticed that, since $\frac{du}{dr}$ is a function of r and θ, we have

$$d\frac{du}{dr} = \frac{d}{dr}\left(\frac{du}{dr}\right) dr + \frac{d}{d\theta}\left(\frac{du}{dr}\right) d\theta,$$

or

$$\frac{d}{dx}\frac{du}{dr} = \frac{d^2u}{dr^2}\frac{dr}{dx} + \frac{d^2u}{drd\theta}\frac{d\theta}{dx}.$$

The values of $\frac{dr}{dx}$ and $\frac{d\theta}{dx}$ are given in equations (3) of the preceding article. Hence, differentiating and substituting, we derive

$$\frac{d^2u}{dx^2} = \frac{d^2u}{dr^2}\cos^2\theta - 2\frac{d^2u}{drd\theta}\frac{\sin\theta\cos\theta}{r} + \frac{du}{dr}\frac{\sin^2\theta}{r} + \frac{d^2u}{d\theta^2}\frac{\sin^2\theta}{r^2} + 2\frac{du}{d\theta}\frac{\sin\theta\cos\theta}{r^2}. \quad . \; . \; (6)$$

Since the effect of putting $\frac{1}{2}\pi - \theta$ in place of θ in equations (1) is to interchange x and y, the expression for $\frac{d^2u}{dy^2}$ may in this example be derived from equation (6) by interchanging $\sin\theta$ and $\cos\theta$ and reversing the sign of $d\theta$. Hence

$$\frac{d^2u}{dy^2} = \frac{d^2u}{dr^2}\sin^2\theta + 2\frac{d^2u}{drd\theta}\frac{\sin\theta\cos\theta}{r} + \frac{du}{dr}\frac{\cos^2\theta}{r} + \frac{d^2u}{d\theta^2}\frac{\cos^2\theta}{r^2} - 2\frac{du}{d\theta}\frac{\sin\theta\cos\theta}{r^2}. \quad . \; . \; (7)$$

Adding (6) and (7) we have

$$\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}=\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}+\frac{1}{r^2}\frac{d^2u}{d\theta^2}. \quad . \quad . \quad . \quad . \quad (8)$$

422. To transform the expression

$$\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2},$$

having given

$$x=\rho\sin\theta\cos\phi, \qquad y=\rho\sin\theta\sin\phi \qquad z=\rho\cos\theta.$$

Regarding x, y, and z as the rectangular coordinates of a point, if we put $r=\rho\sin\theta$, we shall have

$$x=r\cos\phi \qquad \text{and} \qquad y=r\sin\phi;$$

that is r and ϕ, will be the polar coordinates of the projection of the point upon the plane xy. r and z may therefore be regarded as rectangular coordinates of the given point in a plane passing through it and containing the axis of z; hence the polar coordinates of the point in this plane are ρ and θ.

Now by equation (8) Art. 421, we have

$$\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}=\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}+\frac{1}{r^2}\frac{d^2u}{d\phi^2},$$

and

$$\frac{d^2u}{dz^2}+\frac{d^2u}{dr^2}=\frac{d^2u}{d\rho^2}+\frac{1}{\rho}\frac{du}{d\rho}+\frac{1}{\rho^2}\frac{d^2u}{d\theta^2};$$

hence, adding

$$\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}=\frac{d^2u}{d\rho^2}+\frac{1}{\rho}\frac{du}{d\rho}+\frac{1}{r}\frac{du}{dr}+\frac{1}{r^2}\frac{d^2u}{d\phi^2}+\frac{1}{\rho^2}\frac{d^2u}{d\theta^2}.$$

Since r takes the place of y when ρ and θ are the polar coordinates, equation (5) Art. 420, gives

$$\frac{du}{dr}=\frac{du}{d\rho}\sin\theta+\frac{du}{d\theta}\frac{\cos\theta}{\rho}.$$

Whence, substituting and eliminating r, we have

$$\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}=\frac{d^2u}{d\rho^2}+\frac{2}{\rho}\frac{du}{d\rho}+\frac{\cot\theta}{\rho^2}\frac{du}{d\theta}+\frac{1}{\rho^2\sin^2\theta}\frac{d^2u}{d\phi^2}+\frac{1}{\rho^2}\frac{d^2u}{d\theta^2}.$$

423. Examples sometimes arise in which it is known at the outset that u can be expressed as a function of a single variable which is a function of the original variables. Thus, when u is known to be a function of r, and

$$r^2=x^2+y^2+z^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

let it be required to transform the expression

$$\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}.$$

From (1) we have

$$\frac{dr}{dx}=\frac{x}{r},\qquad \frac{dr}{dy}=\frac{y}{r},\qquad \text{and}\qquad \frac{dr}{dz}=\frac{z}{r};$$

whence $\dfrac{du}{dx}=\dfrac{du}{dr}\cdot\dfrac{x}{r}$, $\dfrac{du}{dy}=\dfrac{du}{dr}\cdot\dfrac{y}{r}$, and $\dfrac{du}{dz}=\dfrac{du}{dr}\cdot\dfrac{z}{r}$,

$$\therefore\quad \frac{d^2u}{dx^2}=\frac{x}{r}\frac{d^2u}{dr^2}\frac{dr}{dx}+\frac{du}{dr}\left[\frac{1}{r}-\frac{x}{r^2}\frac{dr}{dx}\right]=\frac{d^2u}{dr^2}\cdot\frac{x^2}{r^2}+\frac{du}{dr}\frac{r^2-x^2}{r^3},$$

$$\frac{d^2u}{dy^2}=\frac{d^2u}{dr^2}\cdot\frac{y^2}{r^2}+\frac{du}{dr}\frac{r^2-y^2}{r^3},\quad\text{and}\quad \frac{d^2u}{dz^2}=\frac{d^2u}{dr^2}\cdot\frac{z^2}{r^2}+\frac{du}{dr}\frac{r^2-z^2}{r^3}.$$

Adding and reducing, we have

$$\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}=\frac{d^2u}{dr^2}+\frac{du}{dr}\cdot\frac{2}{r}.$$

Examples XL.

1. Change the independent variable from x to z in the equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0, \qquad \text{when} \qquad x=\epsilon^z.$$

$$\frac{d^2y}{dz^2}+y=0.$$

2. Change the independent variable from x to y in the equation

$$\frac{d^2y}{dx^2}+2y\left(\frac{dy}{dx}\right)^2=0.$$

$$\frac{d^2x}{dy^2}-2y\frac{dx}{dy}=0.$$

3. Change the independent variable from y to x in the equation

$$(1-y^2)\frac{d^2u}{dy^2}-y\frac{du}{dy}+a^2u=0, \qquad \text{when} \qquad y=\sin x.$$

$$\frac{d^2u}{dx^2}+a^2u=0.$$

4. Change the independent variable from x to z in the equation

$$x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+\frac{a^2}{x^2}y=0, \qquad \text{when} \qquad x=\frac{1}{z}.$$

$$\frac{d^2y}{dz^2}+a^2y=0.$$

5. Change the independent variable from x to z in the equation

$$x^3\frac{d^3y}{dx^3}+3x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0, \qquad \text{when} \qquad z=\log x.$$

$$\frac{d^3y}{dz^3}+y=0.$$

6. Change the independent variable from x to t in the equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0, \qquad \text{when} \qquad x^2 = 4t.$$

$$t\frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 0.$$

7. Given $x = t + t^2$, transform $\frac{d^2u}{dt^2}$ into an expression in which x is the independent variable.

$$\frac{d^2u}{dt^2} = (1 + 4x)\frac{d^2u}{dx^2} + 2\frac{du}{dx}.$$

8. Change the independent variable from x to z in the equation

$$(1 + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 2ax = 0, \qquad \text{when} \qquad z = \log[x + \sqrt{(1 + x^2)}].$$

$$\frac{d^2y}{dz^2} + a(\varepsilon^z - \varepsilon^{-z}).$$

9. Change the independent variable from x to z in the equation

$$(a^2 - x^2)\frac{d^2y}{dx^2} - \frac{a^2}{x}\frac{dy}{dx} + \frac{x^2}{a} = 0, \qquad \text{when} \qquad x^2 + z^2 = a^2.$$

$$a\frac{d^2y}{dz^2} + 1 = 0.$$

10. Change the independent variable from z to x in the equation

$$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} - 1 = (\log z)^2\left[z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz}\right], \qquad \text{when} \qquad z = \varepsilon^{\sin x}.$$

$$\frac{d^2y}{dx^2} + \tan x\frac{dy}{dx} - 1 = 0.$$

11. Change the independent variable from x to t in the equation

$$\frac{d^2y}{dx^2} + 2\frac{\varepsilon^{2x} - \varepsilon^{-2x}}{\varepsilon^{2x} + \varepsilon^{-2x}}\frac{dy}{dx} + \frac{4n^2y}{(\varepsilon^{2x} + \varepsilon^{-2x})^2} = 0, \qquad \text{when} \qquad x = \log\sqrt{\tan t}$$

$$\frac{d^2y}{dt^2} + n^2y = 0.$$

12. Change the independent variable from y to x in the equation

$$\frac{d^3u}{dy^3} - 4\tan y\frac{d^2u}{dy^2} + 2\tan^2 y\frac{du}{dy} = 0, \qquad \text{having given} \qquad \tan y = x.$$

$$(1 + x^2)^2\frac{d^3u}{dx^3} + 2x(1 + x^2)\frac{d^2u}{dx^2} + 2\frac{du}{dx} = 0.$$

13. Change the independent variable from y to x in the equation

$$(1 - y^2)^2\frac{d^2u}{dy^2} - 2y(1 - y^2)\frac{du}{dy} + \frac{2a}{1 - y}u = 0, \qquad \text{when} \qquad y = \frac{\varepsilon^x - \varepsilon^{-x}}{\varepsilon^x + \varepsilon^{-x}}.$$

Employ $dy = (1 - y^2)\, dx.$

$$\frac{d^2u}{dx^2} + a(\varepsilon^{2x} + 1)u = 0.$$

14. Given $x = \varepsilon^\theta$, transform the equation

$$x^3\frac{d^3y}{dx^3} + ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0.$$

Employ equation (4) *Art.* 417.

$$\frac{d^3y}{d\theta^3} + (a - 3)\frac{d^2y}{d\theta^2} + (b - a + 2)\frac{dy}{dx} + cy = 0.$$

15. Show that when $a = 3$, $b = -2$, and $c = 2$, the equation given in the preceding example is equivalent to

$$\left[\frac{d}{d\theta} - 1\right]^2\left[\frac{d}{d\theta} + 2\right]y = 0.$$

16. Given $x = a(1 - \cos t)$ and $y = a(nt + \sin t)$, prove that

$$\frac{d^2y}{dx^2} = -\frac{n\cos t + 1}{a\sin^3 t}.$$

17. Given $x = r\cos\theta$ and $y = r\sin\theta$, y being a function of x, prove that

$$\frac{d^2y}{dx^2} = \frac{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{\left(\cos\theta\frac{dr}{d\theta} - r\sin\theta\right)^3}.$$

18. Given $\quad x\dfrac{d^2y}{dx^2} - \dfrac{x}{y}\left(\dfrac{dy}{dx}\right)^2 + \dfrac{dy}{dx} = 0, \quad$ and $\quad x = y\varepsilon^z,$

prove that
$$y\frac{d^2z}{dy^2} + \frac{dz}{dy} = 0.$$

19. Given $\quad x = \sin\theta \quad$ and $\quad y = \varepsilon^{-\theta}, \quad$ prove that

$$\frac{d^3y}{dx^3} = \frac{\varepsilon^{-\theta}}{\cos^5\theta}[3\sin\theta\cos\theta - \sin^2\theta - 2].$$

20. When $x = \varepsilon^\theta$ prove by means of equation (3), Art. 417, that

$$x^{\frac{3}{2}}\frac{d^2}{dx^2}x^{\frac{1}{2}}\cdot u = \left[\frac{d^2}{d\theta^2} - \frac{1}{4}\right]\cdot u,$$

and verify when $u = \sin x$.

21. When $x = \varepsilon^\theta$, show that

$$\frac{d^2}{dx^2}x^{-3}\frac{d}{dx}x^5\cdot u = \varepsilon^{-\theta}\left[\frac{d}{d\theta} + 5\right]\left[\frac{d}{d\theta} + 1\right]\frac{d}{d\theta}\cdot u.$$

22. By means of the principle deduced in Art. 418, simplify the expression

$$\left[x^{-1}\frac{d}{dx}\right]^2 x^5 \left[x^{-1}\frac{d}{dx}\right]^3.$$

Solution :—

$$\left[x^{-1}\frac{d}{dx}\right]^2 x^5 \left[x^{-1}\frac{d}{dx}\right]^3 = x^{-1}\cdot\frac{d}{dx}x^{-1}\frac{d}{dx}x^3\cdot x\frac{d}{dx}\cdot x^{-1}\frac{d}{dx}x^{-1}\frac{d}{dx},$$

in which the factors between the periods are commutative; hence the expression becomes

$$\frac{d^2}{dx^2}x^{-1}\frac{d}{dx}x^2\frac{d}{dx}x^{-1}\frac{d}{dx} = \frac{d^2}{dx^2}x^{-1}\cdot\frac{d}{dx}x\cdot x\frac{d}{dx}\cdot x^{-1}\frac{d}{dx} = \frac{d^5}{dx^5}.$$

23. Given $u = \varepsilon^{x^2}$ prove that $4\dfrac{d}{dx}x^3\dfrac{d}{dx}xu = x^3\dfrac{d^4}{dx^4}u.$

Solution :—

In this case, $$\frac{d}{dx}u = 2xu\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

hence, by Art. 418,

$$4\frac{d}{dx}x^3\frac{d}{dx}xu = 2\frac{d}{dx}x^3\frac{d^2}{dx^2}u = 2x^3\frac{d^3}{dx^3}xu\,;$$

therefore, by equation (1),

$$4\frac{d}{dx}x^3\frac{d}{dx}xu = x^3\frac{d^4}{dx^4}u.$$

24. Prove that $\left[\dfrac{d}{dx}\right]^n\left[x\dfrac{d}{dx} - n\right]^r y = \left[x\dfrac{d}{dx}\right]^r\dfrac{d^n y}{dx^n}.$

Solution :—

When $x = \varepsilon^\theta$, $\left[x\dfrac{d}{dx} - n\right]^r = \left[\dfrac{d}{d\theta} - n\right]^r;$

hence the symbol in the left hand member is commutative with symbols of the form considered in Art 418; therefore

$$\left[\frac{d}{dx}\right]^n\left[x\frac{d}{dx} - n\right]^r = x^{-n}\left[x\frac{d}{dx} - n\right]^r x^n\left[\frac{d}{dx}\right]^n$$
$$= \varepsilon^{-n\theta}\left[\frac{d}{d\theta} - n\right]^r\varepsilon^{n\theta}\left[\frac{d}{dx}\right]^n,$$

which, by Art. 410, reduces to $\left[\dfrac{d}{d\theta}\right]^r\left[\dfrac{d}{dx}\right]^n.$

25. Given $x = \sqrt{z}$, prove that $\left[\dfrac{1}{x}\dfrac{d}{dx}\right]^n = 2^n\dfrac{d^n}{dz^n};$

and thence find the value of

$$\left[\frac{1}{x}\frac{d}{dx}\right]^n x^2 \varepsilon^{x^2}, \qquad \text{and of} \qquad \left[\frac{1}{x}\frac{d}{dx}\right]^n \log x.$$

See Ex. XII, 18, *and Art.* 84.

$$2^n (x^2 + n)\, \varepsilon^{x^2}, \quad \text{and} \quad (n-1)(n-2)\cdots 1\,(-2)^{n-1} x^{-2n}.$$

26. Given $x = \dfrac{1}{z}$, prove that $\left[x^2 \dfrac{d}{dx}\right]^n = \left[-\dfrac{d}{dz}\right]^n$;

and thence find the value of

$$\left[x^2 \frac{d}{dx}\right]^n \log x, \qquad \text{and of} \qquad \left[x^2 \frac{d}{dx}\right]^n \sin\frac{1}{x}.$$

$$1\cdot 2\cdot 3\cdots(n-1)\,x^n, \quad \text{and} \quad (-1)^n \sin\left\{\frac{1}{x} + \frac{n}{2}\pi\right\}.$$

27. By putting $x = \sqrt{z}$ in the formula proved in example 22, derive the equation

$$\left[\frac{d}{dz}\right]^3 z^{\frac{5}{3}} \left[\frac{d}{dz}\right]^2 = \left[z^{\frac{1}{3}}\frac{d}{dz}\right]^5,$$

and verify for the function z^m.

28. Given $x = r\cos\theta$, and $y = r\sin\theta$, prove that

$$x\frac{du}{dy} - y\frac{du}{dx} = \frac{du}{d\theta}, \qquad \text{and that} \qquad x\frac{du}{dx} + y\frac{du}{dy} = r\frac{du}{dr}.$$

29. If $\xi = x\cos\alpha - y\sin\alpha$, and $\eta = x\sin\alpha + y\cos\alpha$, prove that

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = \frac{d^2u}{d\xi^2} + \frac{d^2u}{d\eta^2}.$$

30. Transform the expression $r^2\dfrac{d^2u}{dr^2} + \dfrac{d^2u}{d\theta^2}$ into a function in which

x and y are the independent variables, having given $x = r\cos\theta$ and $y = r\sin\theta$.

$$r^2\frac{d^2u}{dr^2} + \frac{d^2u}{d\theta^2} = (x^2 + y^2)\left[\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2}\right] - x\frac{du}{dx} - y\frac{du}{dy}$$

31. If $s = \varepsilon^x + \varepsilon^y$, and $t = \varepsilon^{-x} + \varepsilon^{-y}$, prove that

$$\frac{d^2u}{dx^2} + 2\frac{d^2u}{dx\,dy} + \frac{d^2u}{dy^2} = s^2\frac{d^2u}{ds^2} - 2st\frac{d^2u}{ds\,dt} + t^2\frac{d^2u}{dt^2} + s\frac{du}{ds} + t\frac{du}{dt}.$$

32. If $x = a\varepsilon^\theta\cos\phi$, and $y = a\varepsilon^\theta\sin\phi$, prove that

$$y^2\frac{d^2u}{dx^2} - 2xy\frac{d^2u}{dx\,dy} + x^2\frac{d^2u}{dy^2} = \frac{d^2u}{d\phi^2} + \frac{du}{d\theta}.$$

XLI.

Lagrange's Theorem.

424. Let y be an implicit function of the independent variables x and z, satisfying the relation

$$y = z + x\phi(y), \quad \ldots \ldots \quad (1)$$

in which ϕ denotes any function; then, if we have

$$u = f(y), \quad \ldots \ldots \quad (2)$$

u will also be a function of x and z.

If now it be required to develop u in a series involving powers of x, we obtain by the application of Maclaurin's theorem

$$u = u_0 + \left.\frac{du}{dx}\right]_0 x + \left.\frac{d^2u}{dx^2}\right]_0 \frac{x^2}{1\cdot 2} + \cdots, \quad \ldots \quad (3)$$

in which the coefficients are functions of z. We proceed to

transform the derivatives $\frac{du}{dx}$, $\frac{d^2u}{dx^2}$, etc., into expressions in which z is the independent variable, before determining their values when $x = 0$.

From equation (2) we have $du = f'(y)\,dy$, hence by differentiating equation (1) with reference to x and z, we obtain the partial derivatives

$$\frac{du}{dx} = \frac{f'(y)\,\phi(y)}{1 - x\phi'(y)}, \quad \text{and} \quad \frac{du}{dz} = \frac{f'(y)}{1 - x\phi'(y)}; \quad . \quad . \quad (4)$$

whence

$$\frac{du}{dx} = \phi(y)\frac{du}{dz}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

In order to deduce the required expressions for the higher derivatives, we first establish the general theorem—that, when y is a function of x and z, and u and $\psi(y)$ are any functions of y, we have

$$\frac{d}{dx}\psi(y)\frac{du}{dz} = \frac{d}{dz}\psi(y)\frac{du}{dx}. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

To prove this theorem, we have only to perform the differentiations; thus, putting $f(y)$ for u, both members of (6) reduce to

$$\psi'(y)f'(y)\frac{dy}{dx}\frac{dy}{dz} + \psi(y)\frac{d^2u}{dx\,dz}.$$

Substituting in the general theorem (6) the value of $\frac{du}{dx}$ given in equation (5), we have

$$\frac{d}{dx}\psi(y)\frac{du}{dz} = \frac{d}{dz}\phi(y)\cdot\psi(y)\frac{du}{dz}. \quad . \quad . \quad . \quad . \quad (7)$$

Applying the symbol $\frac{d}{dx}$ to equation (5), and reducing the second member by means of equation (7), we find

$$\frac{d^2u}{dx^2} = \frac{d}{dz}[\phi(y)]^2\frac{du}{dz}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Again, applying $\frac{d}{dx}$ to equation (8), we have

$$\frac{d^3u}{dx^3} = \frac{d}{dx}\frac{d}{dz}[\phi(y)]^2\frac{du}{dz} = \frac{d}{dz}\frac{d}{dx}[\phi(y)]^2\frac{du}{dz},$$

and, reducing by equation (7),

$$\frac{d^3u}{dx^3} = \frac{d^2}{dz^2}[\phi(y)]^3\frac{du}{dz};$$

by successive repetitions of this process we obtain in general,

$$\frac{d^nu}{dx^n} = \frac{d^{n-1}}{dz^{n-1}}[\phi(y)]^n\frac{du}{dz}. \quad . \quad . \quad . \quad . \quad . \quad (9)$$

In determining the values which these derivatives assume when $x = 0$, we notice that when $x = 0$ equation (1) gives $y = z$; hence $u_0 = f(z)$, and from (4), $\left.\frac{du}{dz}\right]_{x=0} = f'(z)$. Moreover, since the differentiations indicated in the second member of equation (9) have reference only to z; we may, in this equation, assign to x its value before the differentiations are effected: therefore

$$\left.\frac{du}{dx}\right]_0 = \phi(z)f'(z), \qquad \left.\frac{d^2u}{dx^2}\right]_0 = \frac{d}{dz}\left\{[\phi(z)]^2f'(z)\right\},$$

$$\left.\frac{d^nu}{dx^n}\right]_0 = \frac{d^{n-1}}{dz^{n-1}}\left\{[\phi(z)]^nf'(z)\right\}.$$

Substituting these values in equation (3), we obtain

$$f(y) = f(z) + x\phi(z)f'(z) + \frac{x^2}{1\cdot 2}\frac{d}{dz}\left\{[\phi(z)]^2f'(z)\right\} + \cdots$$
$$+ \frac{x^n}{1\cdot 2\cdots n}\frac{d^{n-1}}{dz^{n-1}}\left\{[\phi(z)]^nf'(z)\right\}.$$

This result is known as *Lagrange's Theorem.*

425. As an application we expand the function

$$y = z + x\varepsilon^y.$$

In this example $f(y) = y \therefore f'(y) = 1$, and $\phi(y) = \varepsilon^y$.

The general term is therefore

$$\frac{x^n}{1\cdot 2\cdots n}\frac{d^{n-1}}{dz^{n-1}}\varepsilon^{nz} = n^{n-1}\varepsilon^{nz}\frac{x^n}{1\cdot 2\cdots n};$$

whence

$$y = z + \varepsilon^z\cdot x + 2\varepsilon^{2z}\cdot\frac{x^2}{1\cdot 2} + 3^2\varepsilon^{3z}\frac{x^3}{1\cdot 2\cdot 3} + \cdots$$

To obtain the development when the function is

$$y = 1 + x\varepsilon^y,$$

we put 1 in place of z in the preceding development. (Compare Ex. XXIII, 14.)

426. When the given relation between x and y is not in the form required for the application of Lagrange's theorem, an algebraic transformation sometimes enables us to make the application. Thus, if we have

$$\log y = xy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

to develop y in powers of x, we put $\log y = y'$; whence we have

$$y = \varepsilon^{y'}, \qquad \text{and (1) becomes} \qquad y' = x\varepsilon^{y'}.$$

The latter equation being in the required form, we have

$$u = y = \varepsilon^{y'}, \qquad f(y') = \varepsilon^{y'}, \qquad \text{and} \qquad \phi(y') = \varepsilon^{y'}.$$

Hence, the general term is

$$\frac{x^n}{1\cdot 2\cdots n}\frac{d^{n-1}}{dz^{n-1}}\varepsilon^{(n+1)z} = \frac{x^n}{1\cdot 2\cdots n}(n+1)^{n-1}\varepsilon^{(n+1)z},$$

and putting $z = 0$, we have

$$y = 1 + x + 3\frac{x^2}{1\cdot 2} + 4^2\frac{x^3}{1\cdot 2\cdot 3} + \cdots$$

427. Lagrange's theorem may, in fact, be applied to $f(y)$ whenever the relation between y and x is of the form

$$y = F[z + x\phi(y)];$$

for, if we put

$$t = z + x\phi(y), \qquad \text{we have} \qquad y = F(t);$$

whence $\quad u = fF(t), \quad$ and $\quad t = z + x\phi F(t).$

Lagrange's theorem is therefore immediately applicable, the functions fF and ϕF taking the place of f and ϕ in the development. Hence, substituting, we have

$$f(y) = fF(z) + x\phi F(z)\frac{dfF(z)}{dz} + \frac{x^2}{1\cdot 2}\frac{d}{dz}\left\{[\phi F(z)]^2\frac{dfF(z)}{dz}\right\} + \cdots$$

This form of the series is called *Laplace's Theorem.*

428. The example in Art. 426 may be regarded as a case of this theorem; for the given equation may be written in the form

$$y = \epsilon^{xy},$$

and we have in this case $f(y) = y$, also

$$F(t) = \epsilon^t, \qquad t = xy, \qquad z = 0, \qquad \text{and} \qquad \phi(y) = y.$$

Both $fF(z)$ and $\phi F(z)$ reduce to ϵ^z, and are identical with $f(z)$ and $\phi(z)$ of Art. 426.

Since Lagrange's theorem is simply Maclaurin's theorem with transformed coefficients, it is always possible to make any series derived by its application convergent by giving to x a value sufficiently small. See Art. 157.

The Development of Functions of Two Independent Variables.

429. Let
$$u = f(x, y), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$
and let it be required to develop $f(x_0 + h, y_0 + k)$ in a series involving powers and products of powers of h and k. Let τ denote any assumed interval of time, and put
$$a = \frac{h}{\tau}, \quad \text{and} \quad b = \frac{k}{\tau}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If now we assume
$$x = x_0 + at \quad \text{and} \quad y = y_0 + bt, \quad . \quad . \quad . \quad (3)$$
t denoting a variable interval of time, u will become a function of t, and we may write
$$u = f(x_0 + at,\ y_0 + bt) = \phi(t) \quad . \quad . \quad . \quad . \quad (4)$$
Putting $t = \tau$ in this equation, we have by equations (2)
$$f(x_0 + h,\ y_0 + k) = \phi(\tau)$$

Developing $\phi(\tau)$ by Maclaurin's theorem, we have
$$\phi(\tau) = \phi(0) + \tau\phi'(0) + \frac{\tau^2}{1 \cdot 2}\phi''(0) + \cdots$$
$$+ \frac{\tau^n}{1 \cdot 2 \cdots n}\phi^n(0) + \frac{\tau^{n+1}}{1 \cdot 2 \cdots (n+1)}\phi^{n+1}(\theta\tau). \quad . \quad . \quad (5)$$

Since $\phi(t) = u$ is a function of x and y, we have
$$\phi'(t) = \frac{du}{dt} = \frac{du}{dx}\frac{dx}{dt} + \frac{du}{dy}\frac{dy}{dt},$$

or, since by equation (3) $\dfrac{dx}{dt} = a$ and $\dfrac{dy}{dt} = b$,

$$\phi'(t) = a\frac{du}{dx} + b\frac{du}{dy}.$$

Hence $$\phi''(t) = \frac{d^2u}{dt^2} = a^2\frac{d^2u}{dx^2} + 2ab\frac{d^2u}{dx\,dy} + b^2\frac{d^2u}{dy^2},$$

and in general

$$\phi^n(t) = \frac{d^nu}{dt^n} = \left[a\frac{d}{dx} + b\frac{d}{dy}\right]^n u. \quad . \quad . \quad . \quad . \quad (6)$$

Putting $t = 0$ in the expressions for $\phi(t)$, $\phi'(t)$, etc., and substituting in (5) we have

$$f(x_0 + h,\ y_0 + k) = f(x_0,\ y_0) + \tau\left[a\frac{du}{dx} + b\frac{du}{dy}\right]_0$$
$$+ \frac{\tau^2}{1\cdot 2}\left[a^2\frac{d^2u}{dx^2} + 2ab\frac{d^2u}{dx\,dy} + b^2\frac{d^2u}{dy^2}\right]_0 + \cdots + R.$$

Substituting for a and b their values from equation (2), and omitting the subscripts since x_0 and y_0 alone appear in the result, we have

$$f(x + h,\ y + k) = f(x,\ y) + h\frac{du}{dx} + k\frac{du}{dy} +$$
$$\frac{1}{1\cdot 2}\left[h\frac{d}{dx} + k\frac{d}{dy}\right]^2 u + \cdots + R. \quad . \quad . \quad (7)$$

In equation (5) the remainder is obtained by putting $t = \theta\tau$, instead of $t = 0$, in the expression for the $(n + 1)th$ derivative, and since $t = \theta\tau$ in equation (3) gives $x = x_0 + \theta h$ and $y = y_0 + \theta k$, we have

$$R = \frac{1}{1\cdot 2 \cdots (n + 1)}\left[h\frac{d}{dx} + k\frac{d}{dy}\right]^{n+1} f(x + \theta h,\ y + \theta k).$$

The Symbolic Expression for the Series.

430. Since the coefficients and indices in the above result follow the law of the exponential series, equation (7) may be written in the symbolic form

$$f(x+h,\, y+k) = \varepsilon^{h\frac{d}{dx} + k\frac{d}{dy}} f(x, y).$$

It follows from Art. 176 that the application of the symbol $\varepsilon^{h\frac{d}{dx}}$ to u is equivalent to changing x to $x + h$. Accordingly the application of the symbol $\varepsilon^{k\frac{d}{dy}}$ to the result thus obtained is equivalent to changing y to $y + k$. The preceding demonstration shows therefore that the symbol $\varepsilon^{h\frac{d}{dx} + k\frac{d}{dy}}$ is equivalent to $\varepsilon^{h\frac{d}{dx}} \cdot \varepsilon^{k\frac{d}{dy}}$, and the form of the series shows that the symbols $\varepsilon^{h\frac{d}{dx}}$ and $\varepsilon^{k\frac{d}{dy}}$ are commutative.

In like manner, were u a function of z, as well as of x and y, the application of the symbol $\varepsilon^{l\frac{d}{dz}}$ to the result obtained above would change z to $z + l$; whence it may be inferred that

$$f(x+h,\, y+k,\, z+l) = \varepsilon^{l\frac{d}{dz}}\, \varepsilon^{h\frac{d}{dx} + k\frac{d}{dy}} f(x, y, z)$$

$$= \varepsilon^{h\frac{d}{dx} + k\frac{d}{dy} + l\frac{d}{dz}} f(x, y, z).$$

Maxima and Minima of Functions of Two Independent Variables.

431. A function u of two independent variables x and y is said to have a ***maximum*** value corresponding to certain values, a and b, of x and y, when after increasing, u begins to decrease, as x and y pass simultaneously through these values, ***whatever***

be the comparative rates with which x *and* y *vary.* A minimum value of u is defined in a similar manner.

It is obvious that a maximum value, as defined above, must continue to be a maximum when either y or x is made constant; hence both $\frac{du}{dx}$ and $\frac{du}{dy}$ must change sign from + to −, as x increases through the value a and y increases through the value b. See Art. 119. In like manner for a minimum both derivatives must change sign from − to +.

The derivatives can change sign only on passing through infinity or zero; it is only when both derivatives pass through zero that criteria founded upon the higher derivatives can be employed for discriminating between maxima and minima. This, however, is the case which most frequently presents itself.

432. Let a and b denote values of x and y determined by the simultaneous equations

$$\frac{du}{dx} = 0, \qquad \text{and} \qquad \frac{du}{dy} = 0. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If we cause x and y to pass through the values a and b with the arbitrary rates

$$\frac{dx}{dt} = \alpha, \qquad \text{and} \qquad \frac{dy}{dt} = \beta, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

u will become a function of t. Hence we shall have

$$\frac{du}{dt} = \frac{du}{dx}\frac{dx}{dt} + \frac{du}{dy}\frac{dy}{dt} = \alpha\frac{du}{dx} + \beta\frac{du}{dy}; \quad . \quad . \quad . \quad (3)$$

therefore, when $x = a$ and $y = b$, $\frac{du}{dt}$ is zero for all values of α and β. Hence, by Art. 133, we shall have a maximum value of u when $\frac{d^2u}{dt^2}$ is negative, and a minimum value when $\frac{d^2u}{dt^2}$ is

positive. Differentiating equation (3), the arbitrary rates α and β being assumed to be constant, we have

$$\frac{d^2u}{dt^2} = \alpha^2 \frac{d^2u}{dx^2} + 2\alpha\beta \frac{d^2u}{dx\,dy} + \beta^2 \frac{d^2u}{dy^2}. \quad . \quad . \quad . \quad (4)$$

Putting $A = \left.\frac{d^2u}{dx^2}\right]_{a,b}$, $B = \left.\frac{d^2u}{dx\,dy}\right]_{a,b}$, and $C = \left.\frac{d^2u}{dy^2}\right]_{a,b}$,

we shall therefore have a maximum or a minimum when the expression

$$A\alpha^2 + 2B\alpha\beta + C\beta^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

retains the same signs for all values of α and β. Writing this expression in the equivalent form

$$\frac{\alpha^2}{C}\left[\left(C\frac{\beta}{\alpha} + B\right)^2 + (AC - B^2)\right], \quad . \quad . \quad . \quad . \quad (6)$$

it is obvious that this condition will be fulfilled when we have

$$AC - B^2 > 0.$$

This is known as *Lagrange's condition.* When this condition is fulfilled, A and C have like signs, and the sign of the expression for $\frac{d^2u}{dt^2}$ is, for all values of the ratio $\frac{\beta}{\alpha}$, the same as that of A and C. Hence the value of u will be a ***minimum*** when A and C are ***positive,*** and will be a ***maximum*** when A and C are ***negative.***

433. As an illustration, let the function be

$$u = a^2 + b^2 - x^2 - y^2 - 2b\sqrt{(a^2 - y^2)};$$

whence $\quad \frac{du}{dx} = -2x \quad$ and $\quad \frac{du}{dy} = -2y + \frac{2by}{\sqrt{(a^2-y^2)}}.$

These derivatives vanish when $x = 0$ and $y = 0$. We have also

$$\frac{d^2u}{dx^2} = -2, \qquad \frac{d^2u}{dx\,dy} = 0, \qquad \frac{d^2u}{dy^2} = -2 + 2\frac{ba^2}{(a^2 - y^2)^{\frac{3}{2}}},$$

therefore, when $y = 0$,

$$A = -2 \qquad B = 0 \qquad C = -2 + 2\frac{b}{a};$$

whence
$$AC - B^2 = 4\frac{a-b}{a}.$$

Hence, if $a > b$, Lagrange's condition is satisfied, and u is a maximum for $x = 0, y = 0$, since A and C are negative. If, on the other hand, $a < b$, Lagrange's condition is not fulfilled, since $AC - B^2$ is negative, consequently there is neither a maximum nor a minimum. In this case, expression (5) becomes

$$-2\left[\alpha^2 - \frac{b-a}{a}\beta^2\right],$$

which is negative when $\frac{\beta^2}{\alpha^2} < \frac{a}{b-a}$, and positive when $\frac{\beta^2}{\alpha^2} > \frac{a}{b-a}$.

434. When $A = 0$, $B = 0$, and $C = 0$, the expression for $\frac{d^2u}{dt^2}$ vanishes for all values of α and β; hence (see Art. 138), it becomes necessary to examine the higher derivatives.

The general form of the mth derivative is

$$\frac{d^m u}{dt^m} = \left(\alpha\frac{d}{dx} + \beta\frac{d}{dy}\right)^m u. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This expression when expanded involves all the partial derivatives of the mth order with reference to x and y; hence it will vanish for all values of α and β only when all the partial derivatives of the mth order vanish. Now let n denote the order of

the lowest derivative that does not vanish. It is shown in Art. 138 that there can be a maximum or a minimum only when n is an even number, and it is further necessary that the expression for $\dfrac{d^n u}{dt^n}$ shall retain the same sign for all values of α and β. This will be the case only when all the values of $\dfrac{\beta}{\alpha}$ derived from the equation formed by putting the expanded expression for this derivative equal to zero are impossible.

435. If we have $AC - B^2 = 0$ when A, B, and C do not vanish, expression (6), Art. 432, has the same sign as C for all values of α and β, except those which make

$$C\frac{\beta}{\alpha} + B = 0; \qquad \text{that is,} \qquad \frac{\beta}{\alpha} = -\frac{B}{C}.$$

When this case presents itself it is necessary to examine the higher derivatives for this particular value of the ratio. The illustrative example in Art. 433 furnishes an instance of this case. For, if $a = b$, the expression for the second derivative is $-2\alpha^2$ indicating a maximum except when $\alpha = 0$. Now when $\alpha = 0$ equation (1) of Art. 434 becomes

$$\frac{d^m u}{dt^m} = \beta^m \frac{d^m z}{dy^m},$$

and from the value of $\dfrac{d^2 u}{dy^2}$ already determined we derive

$$\frac{d^3 u}{dy^3} = 6ba^2 y\,(a^2 - y^2)^{-\frac{5}{2}}, \quad \frac{d^4 u}{dy^4} = 6ba^2\,[(a^2 - y^2)^{-\frac{5}{2}} + 5y^2\,(a^2 - y^2)^{-\frac{7}{2}}].$$

Hence, when $y = 0$, the third derivative vanishes and the fourth derivative becomes $6b\beta^4 a^{-3}$, a positive quantity, indicating the existence of a minimum when $\alpha = 0$; but since we have a maximum for all other values of α it is obvious that the value of the function is not in this case a true maximum.

436. When $\frac{du}{dx}$ and $\frac{du}{dy}$ have a common factor, by putting this factor equal to zero, we obtain a relation between x and y, and it is obvious that these derivatives will vanish for all values of x and y which satisfy this relation. Such values do not however correspond to true maxima or minima values of u. For, when x and y vary, *subject to this relation*, it is evident that $\frac{du}{dt}$ will remain zero, and consequently that u will remain constant; that is, it is possible for x and y to vary so as neither to increase nor diminish the value of u, therefore u is neither a maximum nor a minimum. As an illustration of this case, see examples 13 and 15 below.

Examples XLI.

1. Given $y = z + x\varepsilon^{py}$, expand ε^{my} in powers of x.

$$\varepsilon^{my} = \varepsilon^{mz} + xm\varepsilon^{(p+m)z} + \frac{x^2}{1\cdot 2} m(2p+m)\,\varepsilon^{(2p+m)z} + \cdots$$

$$+ \frac{x^n}{1\cdot 2\cdots n} m(np+m)^{n-1}\varepsilon^{(np+m)z} + \cdots$$

2. Given $y = a + xy^2$, expand y in powers of x.

$$y = a + a^2x + 6a^5\frac{x^2}{1\cdot 2} + 9\cdot 8a^7\frac{x^3}{1\cdot 2\cdot 3}$$

$$+ 12\cdot 11\cdot 10\,a^9\frac{x^4}{1\cdot 2\cdot 3\cdot 4} + \cdots$$

3. Given $y = z + x\frac{y^2-1}{2}$, expand y in powers of x.

$$y = z + \tfrac{1}{2}(z^2-1)x + \tfrac{1}{2}z(z^2-1)x^2 + \tfrac{1}{8}(5z^4-6z^2+1)x^3 + \cdots \quad .$$

4. Given $y = \frac{\pi}{4} + x \sin y$, to expand $\cos y$ in powers of x.

$$\cos y = \frac{1}{\sqrt{2}} - \frac{x}{2} - \frac{3x^2}{4\sqrt{2}} - \frac{x^3}{3} \cdots - \frac{x^n}{1\cdot 2 \cdots n} \frac{d^{n-1}}{dz^{n-1}} (\sin z)^{n+1} \Big]_{z=\frac{\pi}{4}}.$$

5. Given $y = a + by^n$, expand y in powers of b.

$$y = a + a^n b + 2na^{2n-1} \frac{b^2}{1\cdot 2} + 3n(3n-1)\, a^{3n-2} \frac{b^3}{1\cdot 2\cdot 3} + \cdots$$

6. Given $x^5 + 4x + 2 = 0$, determine the value of x.

$$x = -\tfrac{1}{2} + \frac{1}{2^7} - \frac{5}{2^{13}} + \frac{35}{2^{19}} - \cdots$$

7. Given $y = a + xy^3$, expand y^3 in powers of x.

$$y^3 = a^3 + 3a^5 x + 8a^7 \cdot \frac{3x^2}{1\cdot 2} + 11\cdot 10 a^9 \frac{3x^3}{1\cdot 2\cdot 3} + \cdots$$

8. Given $y = \varepsilon + x \log y$, expand y in powers of x.

$$y = \varepsilon + x + \frac{2}{\varepsilon} \cdot \frac{x^2}{1\cdot 2} + \frac{3}{\varepsilon^2} \cdot \frac{x^3}{1\cdot 2\cdot 3} - \frac{4}{\varepsilon^3} \cdot \frac{x^4}{1\cdot 2 \cdots 4} \cdots$$

9. Given $y = x\varepsilon^{-y}$, expand $\sin(\alpha + y)$ in powers of x.

Solution :—

The coefficient of $\frac{x^n}{1\cdot 2 \cdots n}$ is $\frac{d^{n-1}}{dz^{n-1}} [\varepsilon^{-nz} \cos(\alpha + z)]$;

and by equation (2), Art. 410, we have

$$\frac{d^{n-1}}{dz^{n-1}} \varepsilon^{-nz} \cos(\alpha + z) = \varepsilon^{-nz} \left(\frac{d}{dz} - n\right)^{n-1} \cos(\alpha + z).$$

Now $\left(\frac{d}{dz} - n\right) \cos(\alpha + z) = -\sin(\alpha + z) - n\cos(\alpha + z)$,